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Abstract. We prove a strong law of large numbers for directed last passage times in an independent but inhomogeneous exponential environment. Rates for the exponential random variables are obtained from a discretisation of a speed function that may be discontinuous on a locally finite set of discontinuity curves. The limiting shape is cast as a variational formula that maximises a certain functional over a set of weakly increasing curves.

Résumé. On montre une loi des grands nombres pour les temps de dernier passage dirigé dans un environnement indépendant mais inhomogène et exponentiel. Les taux des variables exponentielles sont obtenues a partir d'une discretisation d'une fonction de vitesse macroscopique qui pourrait être discontinue sur un ensemble localement fini des courbes de discontinuité. La forme à la limite est déterminée par une formule des variations qui maximise une certaine fonctionnel sur un ensemble des courbes faiblement croissantes. Dans le processus, on obtient des propriétés de continuité pour la forme limite.

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1. Introduction

We consider a model of directed last passage growth model in two dimensions, where each lattice site (i,j) of $\mathbb{Z}_+^2$ is given a random weight $\tau_{i,j}$ according to some background measure $\mathbb{P}$.

Given lattice points $(a,b),(u,v) \in \mathbb{Z}_+^2$, $\Pi_{(a,b),(u,v)}$ is the set of lattice paths $\pi = \{(a,b) = (i_0,j_0),(i_1,j_1),\dots,(i_p,j_p) = (u,v)\}$ whose admissible steps satisfy

$$(i_\ell, j_\ell) - (i_{\ell-1}, j_{\ell-1}) \in \{(1,0),(0,1)\}. \tag{1.1}$$

If $(a,b) = (0,0)$ we simply denote this set by $\Pi_{u,v}$.

For $(u,v) \in \mathbb{Z}_+^2$ and $n \in \mathbb{N}$ denote the *last passage time*

$$G_{(a,b),(u,v)} = \max_{\pi \in \Pi_{(a,b),(u,v)}} \Big\{ \sum_{(i,j)\in\pi} \tau_{i,j} \Big\}. \tag{1.2}$$

Again, if $(a,b) = (0,0)$ and no confusion arises, we simply denote $G_{(0,0),(u,v)}$ with $G_{u,v}$. In the homogeneous setting, $\{\tau_{i,j}\}_{(i,j)\in\mathbb{Z}_2^+}$ are i.i.d. under $\mathbb{P}$ and standard subadditivity arguments give the existence of a point-to-point scaling limit

$$\lim_{n\to\infty} \frac{G_{\lceil nx\rceil,\lceil ny\rceil}}{n} = g_{pp}(x,y).$$

Generic properties of $g_{pp}(x,y)$ have been obtained in [26], that are universal under some mild conditions on the distribution of $\tau_{i,j}$. In [6], a distributional limit to a Tracy-Widom law was proven for passage times 'near the edge', i.e. for passage times in thin rectangles of order $n \times n^a$. It is expected that several properties of the last passage models hold irrespective of the distribution of $\tau_{i,j}$; these include the fluctuation exponent of $G_{\lceil nx\rceil,\lceil ny\rceil}$, limiting laws and fluctuations of the maximal path around its macroscopic direction.

In first passage percolation (FPP) processes, existence of limiting constants can be traced back to [13, 22, 25] and several techniques can be transferred to the super-additive directed LPP setting. For last passage models with general weight distributions, see the survey [27]. As far as the law of large numbers goes, a universal approach, under only some moment assumptions on the distribution of $\tau_{i,j}$, has been developed in [19, 28–30], where the limiting shape is given in terms of variational formulas and Busemann functions.

When the environment $\tau_{i,j} \sim$ Exp(1), the last passage model is one of the exactly solvable models of the KPZ class (see [10] for a survey). The strong law of large numbers in the exponential model is explicitly computed in [33]

$$\lim_{n\to\infty} \frac{G_{\lceil nx\rceil,\lceil ny\rceil}}{n} = \gamma(x,y) = (\sqrt{x}+\sqrt{y})^2, \quad \mathbb{P}\text{-a.s.} \tag{1.3}$$

In this article we derive the limiting constant for a sequence of scaled last passage times on the lattice. The passage times themselves are coupled through a common realization of exponential random variables. However, the rates of these random variables will be chosen according to a discrete approximation of a macroscopic function

$$c : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+.$$

Consider the lattice corner $\mathbb{Z}_+^2$. The environment $\tau = \{\tau_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ is a collection of i.i.d. exponential random variables of rate 1. For any $n \in \mathbb{N}$ we alter the rate of each of these random variables by a scalar multiplication using the macroscopic speed function $c(x,y)$. Namely, define

$$r_{i,j}^{(n)} = c\Big(\frac{i}{n}, \frac{j}{n}\Big)^{-1}, \qquad (i,j) \in \mathbb{Z}_+^2, \tag{1.4}$$

and define n-scaled, inhomogeneous environment by

$$\tau_{i,j}^{(n)} = r_{i,j}^{(n)} \tau_{i,j}^{n}. \tag{1.5}$$

The rate of the exponential random variable $\tau_{i,j}^{(n)}$ is now determined by the scalar $c\big(\frac{i}{n}, \frac{j}{n}\big)$. On each site the rate is completely determined by the speed function $c(\cdot,\cdot)$. We indicate the corresponding exponential 1 random variable as $\tau_{i,j}^{n}$.

For $(u,v) \in \mathbb{Z}_+^2$ and $n \in \mathbb{N}$ denote the last passage time

$$G_{u,v}^{(n)} = \max_{\pi\in\Pi_{u,v}} \Big\{ \sum_{(i,j)\in\pi} r_{i,j}^{(n)} \tau_{i,j}^{n} \Big\} = \max_{\pi\in\Pi_{u,v}} \Big\{ \sum_{(i,j)\in\pi} \tau_{i,j}^{(n)} \Big\}. \tag{1.6}$$

We impose several conditions on the function $c(x,y)$ and they are described in Section 2. For the moment we emphasise that for any compact set $K \subseteq \mathbb{R}_+^2$ there exist finite constant m_K and M_K such that

$$m_K \le c(x,y) \le M_K \quad \text{ for all } (x,y) \in K$$

and there are a finite number (that depends on K) of discontinuity curves of the function $c(x,y)$. These are to avoid degeneracies: If $c(x,y)$ can take the value 0 then the environment could take the value ∞ which leads to trivial passage times. If $c(x,y)$ can be infinity, that region of space will never be explored by a path. If the discontinuities have an accumulation point, then no finite descritisation of $c(x,y)$ can capture that.

We prove a strong law of large numbers for $n^{-1}G^{(n)}_{\lceil nx\rceil,\lceil ny\rceil}$. The limiting last passage constant $\Gamma_c(x,y)$ has a variational characterisation that naturally leads to a continuous version of a last passage time model (see Theorem 2.6).

1.1. Inhomogeneous growth models

We are concerned with directed last passage percolation on the lattice in a discontinuous environment; weights $\omega_{i,j}$ at each site (i,j) are exponentially distributed but with different rates that depend on their position. Similar arguments can be repeated when the environment comes from geometrically distributed weights, and in this case the inhomogeneity will be captured by changing the values of p, the probability of success of the geometric weight. Such models do not have the super-additivity properties that guarantee the existence of a macroscopic shape, so other techniques must be utilised to first show existence of macroscopic limits and then compute a formula for them.

Several inhomogeneous models of last passage percolation exist, each one with different ways of assigning rates (or weights in general). One way is to fix two positive sequences $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}$ to assign to site (i,j) an exponential weight $\omega_{i,j}$ with rate a_i+b_j. Laws of large numbers for the last passage time for these model was obtained in [37] when a_i where i.i.d. and b_j constant, and then generalised in [16]. The model enjoys several aspects of integrability, and large deviations from the shape were obtained in [17]. When admissible steps are not restricted to just e_1, e_2, [20] studies an inhomogeneous model which generalises the one introduced in [35] and obtain explicit distributional limits for fluctuations of the passage time.

Inhomogeneities defined via the speed function have been already considered in the literature. When the speed function is continuous, [32] showed the law of large numbers for the passage times and convergence of the microscopic maximal paths to a continuous curve conditioned on uniqueness of the macroscopic maximiser. A model for which the speed function is $c(x,y) = r\mathbb{1}\{x=y\} + \mathbb{1}\{x \neq y\}$ was introduced in [23] and [24] and the law of large numbers was studied in [36] and it was shown that for small values of r the LLN disagrees with that of the 1-homogeneous model. This is the slow bond problem, for which the conjecture that the LLN was different from the i.i.d. model for all $r<1$ was only verified recently in [5].

When the discontinuity curves of $c(x,y)$ was a locally finite set of lines of the form $\{y = x+b_i\}_{i\in\mathbb{N}}$, the law of large numbers limits was obtained in [18] and an explicit limit for the shape function was obtained in the case of the two-phase model with $c(x,y) = r_1\mathbb{1}\{x \le y\} + r_2\mathbb{1}\{x > y\}$. In this case a flat edge was observed for the limiting shape function. Flat edges appear very commonly in first passage percolation (e.g. [15]) but for directed LPP in an i.i.d. environment the only known examples are in discrete environments with percolating maximum [14].

A first passage (unoriented) percolation two-phase model was studied in [1], where the edge-weight distribution was different to the left and right half-planes and in certain cases proved the creation of a 'pyramid' in the limiting shape, i.e. a polygonal segment with a point of non-differentiability at the peak. Indeed, away from an i.i.d. setting many different phenomena can occur. [21] showed that any compact set with lattice symmetries can be obtained as a limit shape of a stationary FPP process. For example, an octagon has been obtained as a limit shape in [2], where also geodesics in the direction of corners (points of non-differentiability) were studied. It is more difficult to obtain polygonal shapes and having points of non differentiability in an LPP setting when the environment is independent and has a continuous distribution, but with the speed functions we are considering here it is possible. You can find two worked-out examples in the v1-Arxiv version of this paper where we show how flat edges, points of non-differentiability, and non-convex limit shapes naturally arise.

In [8] the law of large numbers for directed last passage percolation was extended when the set of discontinuity curves for $c(x,y)$ was a locally finite set of piecewise Lipschitz strictly increasing curves. A PDE approach was used, bypassing the usual techniques of the totally asymmetric simple exclusion process (TASEP) particle systems, used in the earlier articles. In general, several models of percolation with inhomogeneities can be understood by their corresponding particle systems with inhomogeneities [3, 4, 9, 11, 12, 18, 31]. Recently, in [7] a totally asymmetric particle with blockage with spatial inhomogeneities was studied and limiting Tracy-Widom laws were obtained and led to a new kind of a percolation model.

For directed nearest neighbour LPP the corresponding particle system is TASEP. The coupling with TASEP was utilised for example in [18, 32, 36] to obtain results about hydrodynamic limits of the particle current and density, together with the results for the last passage times. In the present we completely avoid the particle system interpretation and focus on the geometrical aspects of the corner growth model.

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1.3. Organisation of the paper

In Section 2 we describe the main theorems. First we state the law of large numbers limit for the passage time (1.6). This is Theorem 2.6. The limiting shape function, denoted by $\Gamma(x,y)$ comes in the form of a variational formula, where a functional is maximised over a set of suitable functions. Continuity properties of Γ are proved in Section 3. The proof of the law of large numbers is in Section 4.

1.4. Commonly used notation

If a variable τ follows the exponential distribution with parameter $r > 0$ this means $\mathbb{P}\{\tau > t\} = e^{-rt}$, in other words r is the rate.

Bold-face letters (e.g. $\mathbf{v}$) indicate two dimensional vectors (e.g. $\mathbf{v} = (v_1, v_2)$). In particular, the letter $\mathbf{x}$ is reserved for denoting two-dimensional curves; often we write $\mathbf{x}(s) = (x_1(s), x_2(s))$ to emphasise that the curve is parametrised and seen as a function. Inequalities of vectors $\mathbf{v} \le \mathbf{w}$ or $(v_1, v_2) \le (w_1, w_2)$ means the inequality holds coordinate-wise. For a vector $\mathbf{v} = (v_1, v_2)$, we denote by $\lfloor \mathbf{v} \rfloor = (\lfloor v_1 \rfloor, \lfloor v_2 \rfloor)$.

Without any special mention, when we write $\| \cdot \|$ we mean $\| \cdot \|_\infty$ unless explicitly referring to a different norm. For any continuous function g we denote its modulus of continuity by ω_g and we assume

$$\|g(z_1) - g(z_2)\|_\infty \le \omega_g(\|z_1 - z_2\|_\infty).$$

In the sequence we use the fact that ω_g is continuous at 0 and that $\omega_g(0) = 0$ without particular mention.

For any set $A \subseteq \mathbb{R}_2^+$, we denote the multiplication $nA = \{(nx, ny) : (x, y) \in \mathbb{R}_+^2\}$ and the floor $\lfloor nA \rfloor = \{(\lfloor nx \rfloor, \lfloor ny \rfloor) : (nx, ny) \in nA\}$. Similarly we use the notation for the ceiling $\lceil nA \rceil$.The topological interior of the set is denoted by $\mathrm{int}(A)$. For vectors $\mathbf{v}, \mathbf{w}$, $\mathbf{v} \le \mathbf{w}$, we denote by $R(\mathbf{v}, \mathbf{w})$ the rectangle with south-west corner $\mathbf{v}$ and north-east corner $\mathbf{w}$.

Letter G is reserved for last passage times. Often we use the notation G_A to denote the last passage time in the set A, which is the maximum weight that can be collected on up-right paths that lie in the set A. If no such paths exist, $G_A = 0$.

2. Model and results

At this point, we state the technical conditions on $c(x, y)$ that we are imposing. There will be no special mention to these in the sequence, unless absolutely necessary. We explain why these assumptions are used after the statement of Theorem 2.6.

We assume the speed function $c(x, y)$ satisfies the following two assumptions:

Assumption 2.1. *[Discontinuity curves of $c(x, y)$] Function $c(x, y)$ is discontinuous on a (potentially) countable set of curves $H_c = \{h_i\}_{i \in \mathcal{I}}$ satisfying the following properties*

(1) h_i is either a linear segment or strictly monotone.
(2) If h_i is not a vertical line segment, it can be viewed as a graph

$$h_i : [z_i, w_i] = \mathrm{Dom}(h_i) \to \mathbb{R},$$

(3) If h_i is strictly increasing, then
   - *(a) h_i is $C^1((z_i, w_i), \mathbb{R})$. At the boundary points z_i, w_i the derivative may take the value $\pm\infty, 0$.*
   - *(b) The equation $h_i'(s) = 0$ has finitely many solutions in $[z_i, w_i]$.*

(4) If h_i is strictly decreasing, then h_i is continuous.
(5) There are finitely many curves h_i in any compact set $K \subseteq \mathbb{R}_+^2$, satisfying $(1) - (4)$. Equivalently, accumulation points of different curves $\{h_j\}_j$ are not allowed.

The discontinuity curves $\{h_i\}_{i \in \mathcal{I}}$ separate $\mathbb{R}_+^2$ into open regions in which $c(x, y)$ is assumed continuous. The number of regions is finite in any compact set of $\mathbb{R}_+^2$. Denote the set of regions by $\mathcal{Q}$.

There are two types of points on these discontinuity curves,

1. (Interior points) These are points $\mathbf{w}$ that belong on a single discontinuity curve h_i. For any point $\mathbf{w}$ of this form, we can find an $\varepsilon > 0$ so that h_i partitions $B(\mathbf{w}, \varepsilon)$ in to three disjoint sets, $U_{\varepsilon, \mathbf{w}}$ (above h_i), $L_{\varepsilon, \mathbf{w}}$ (below h_i) and $(h_i \cap B(\mathbf{w}, \varepsilon))$.
2. (Intersection/terminal points) These are points $\mathbf{w}$ that either belong on more than one discontinuity curve or they are terminal for h_i. There are finitely many of these points in any compact set.

Assumption 2.2. *[Further properties of $c(x, y)$]*

1. *$c(x, y)$ is continuous on any $Q \in \mathcal{Q}$, lower-semicontinuous on $\mathbb{R}_+^2$, that further satisfies the following stability assumption:*
   For every $i \in \mathcal{I}$ and interior point $\mathbf{w} \in h_i$, there exists $\varepsilon = \varepsilon(i, \mathbf{w}) > 0$ so that for all $\mathbf{y} \in B(\mathbf{w}, \varepsilon) \cap h_i$ there exists open set $Q_{i,\mathbf{w}} \in \{L_{\varepsilon,\mathbf{w}}, U_{\varepsilon,\mathbf{w}}\}$, so that for any sequence $\mathbf{z}_n \in Q_{i,\mathbf{w}} \cap B(\mathbf{w}, \varepsilon)$ with $\mathbf{z}_n \to \mathbf{y}$,

$$\lim_{\mathbf{z}_n \to \mathbf{y}} c(\mathbf{z}) = c(\mathbf{y}). \tag{2.1}$$

2. *For any compact set $K \subset \mathbb{R}_+^2$, there exist two constants $r_{\text{low}}^{(K)} > 0$ and $r_{\text{high}}^{(K)} < \infty$, so that*

$$r_{\text{low}}^{(K)} \le c(x,y) \le r_{\text{high}}^{(K)}, \quad \forall (x,y) \in K.$$

Remark 2.3. *Assumption 2.2, (1) gives by a standard compactness argument that if $c(x,y)$ is never continuous on h_i then it must be that in a strip around h_i the values of $c(x,y)$ on one of the incident regions is always smaller than the values in all other incident regions. This is consistent with assumption F3, equation (1.12) in [8]. Lower semi-continuity of $c(x,y)$ implies that the limiting value in (2.1) is the smallest of all possible limits on sequences that approach y. However, the assumption of [8] that $c(x,y)$ is (at least locally) Lipschitz is now removed.*

Fix an (x,y) in $\mathbb{R}_+^2$ and a speed function $c(\cdot,\cdot)$. Define the function $\Gamma_c(x,y)$ via the variational formula

$$\Gamma_c(x,y) = \sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} \left\{ \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s),x_2(s))}\,ds \right\}, \tag{2.2}$$

where $\gamma(x,y) = (\sqrt{x}+\sqrt{y})^2$ is the last-passage constant in a homogeneous rate 1 environment, $\mathbf{x}(s) = (x_1(s),x_2(s))$ denotes a path in $\mathbb{R}^2$ and set

$$\begin{aligned}\mathcal{H}(x,y) = \{\mathbf{x} \in C([0,1],\mathbb{R}_+^2) : \mathbf{x} \text{ is piecewise } C^1, \mathbf{x}(0) = (0,0), \mathbf{x}(1) = (x,y),\\ \mathbf{x}'(s) \in \mathbb{R}_+^2 \text{ wherever the derivative is defined}\}.\end{aligned}$$

When the speed function $c(x,y) = r$ constant, we can immediately compute

$$\begin{aligned}\Gamma_r(x,y) &= \sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s),x_2(s))}\,ds = \frac{1}{r}\sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} \int_0^1 \gamma(\mathbf{x}'(s))\,ds\\ &\le \frac{1}{r}\sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} \gamma\Big(\int_0^1 x_1'(s)\,ds, \int_0^1 x_2'(s)\,ds\Big), \quad \text{by Jensen's inequality since } \gamma \text{ is concave}\\ &= \frac{1}{r}\gamma(x,y) \le \Gamma_r(x,y).\end{aligned}$$

The last inequality follows from the fact that the straight line from 0 to (x,y) is an admissible candidate maximiser for (2.2). The calculation shows two things that we use freely in the sequence, namely

1. Straight lines are optimisers of (2.2) in homogeneous (constant) regions of $c(x,y)$. In fact, because γ is strictly concave, it is easy to show that the straight line will be the unique maximiser. We refer to this fact as 'Jensen's inequality' in the sequence.
2. $\Gamma_r(x,y)$ corresponds to the limiting shape function for last passage times in a homogeneous $\text{Exp}(r)$ environment.

Two more properties of Γ_c can be immediately obtained:

(1) (Independence from parametrization) For any $c > 0$, $\gamma(cx,cy) = c\gamma(x,y)$ so the value of the integral

$$I(\mathbf{x}) = \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s),x_2(s))}\,ds \tag{2.3}$$

is independent of the parametrisation we choose for the curve $\mathbf{x}$.

(2) (Superadditivity) Define $\Gamma_c(x,y) := \Gamma_c((0,0),(x,y))$ and similarly define Γ_c from any starting point (a,b) to any terminal point (x,y), $(x,y) \ge (a,b)$ by

$$\Gamma_c((a,b),(x,y)) = \sup_{\mathbf{x}(\cdot)\in\mathcal{H}((a,b),(x,y))} \left\{ \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s),x_2(s))}\,ds \right\}, \tag{2.4}$$

where

$$\begin{aligned}\mathcal{H}((a,b),(x,y)) = \{\mathbf{x} \in C([0,1],\mathbb{R}_+^2) : \mathbf{x} \text{ is piecewise } C^1, \mathbf{x}(0) = (a,b), \mathbf{x}(1) = (x,y),\\ \mathbf{x}'(s) \in \mathbb{R}_+^2 \text{ wherever the derivative is defined}\}.\end{aligned}$$

Then, for any $(a,b) \le (z,w) \le (x,y)$ we have

$$\Gamma_c((a,b),(x,y)) \ge \Gamma_c((a,b),(z,w)) + \Gamma_c((z,w),(x,y)). \tag{2.5}$$

In this respect, function Γ_c behaves like a 'macroscopic last passage time' and the first theorem shows that it is a continuous function.

Theorem 2.4. *[Continuity properties of Γ.] Let $c(x,y)$ satisfy Assumptions 2.1 and 2.2. Fix (a,b) and $(x,y) \in \mathbb{R}_+^2$. Then, for any $\varepsilon > 0$ there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ so that*

1. *If the boundary of the rectangle $R((a,b),(x,y))$ contains no discontinuity segment of the function c then for all $\delta_1, \delta_2, \delta_3, \delta_4 \in (-\delta_0, \delta_0)$, we have*

$$|\Gamma_c((a+\delta_1, b+\delta_2),(x+\delta_3, y+\delta_4)) - \Gamma_c((a,b),(x,y))| < \varepsilon. \tag{2.6}$$

2. *If the boundary of the rectangle $R((a,b),(x,y))$ contains a discontinuity segment of the function c then for all $\delta_1, \delta_2, \delta_3, \delta_4 \in (0, \delta_0)$, we have*

$$|\Gamma_c((a-\delta_1, b-\delta_2),(x+\delta_3, y+\delta_4)) - \Gamma_c((a,b),(x,y))| < \varepsilon. \tag{2.7}$$

Remark 2.5. *The above theorem says that in general continuity of $\Gamma_c(x,y)$ cannot be guaranteed on points (x,y), if c has horizontal or vertical lines as discontinuity curves and $(a,b),(x,y)$ belong to them. If this is the case, then in general we cannot approximate the terminal point (x,y) from below and the starting point (a,b) from above. This is a purely geometric issue and it has to do with admissible macroscopic paths following these discontinuity curves in order to benefit from larger weights.*

The counterexample that illuminates this point is the following: consider for $\varepsilon > 0$,

$$c_\varepsilon(x,y) = \begin{cases} 1, & \text{for } (x,y) \in R((1,1),(2,2)) \\ \varepsilon, & \text{for } (x,y) \notin R((1,1),(2,2)). \end{cases}$$

Because of lower-semicontinuity, c_ε on the boundary of the rectangle takes the value ε. As such, $\Gamma_{c_\varepsilon}((1,1),(2,2)) = 2\varepsilon^{-1}$ by following the boundary of $R((1,1),(2,2))$, while for any $\delta > 0$, $\Gamma_{c_\varepsilon}((1+\delta, 1+\delta),(2-\delta, 2-\delta)) \le 4$. For $\varepsilon < 1/2$, we cannot have continuity.

This example also justifies why we are using ceilings $\lceil \quad \rceil$ for the endpoints of passage times; we need to be certain that in the discrete lattice maximal paths can take advantage of lower weights on vertical or horizontal discontinuities that could have been not felt, if the discretisation of $c(x,y)$ was not allowing the path to explore the slow region and end in $(\lfloor nx \rfloor, \lfloor ny \rfloor)$.

In the next theorem we obtain Γ_c in (2.2) as the law of large number of the microscopic last passage time (1.6).

Theorem 2.6. *Recall (1.6). Let $c(x,y)$ a macroscopic speed function which satisfies Assumptions 2.1 and 2.2, and let $(x,y) \in \mathbb{R}_+^2$. Then we have the scaling limit*

$$\lim_{n\to\infty} n^{-1} G^{(n)}_{\lceil nx \rceil, \lceil ny \rceil} = \Gamma_c(x,y) \quad \mathbb{P}-a.s. \tag{2.8}$$

Remark 2.7 (The conditions on the discontinuity curves)**.** *In [8] the discontinuity curves are assumed strictly monotone, outside of compact set. As such, when viewed as graphs of continuous functions, they are differentiable almost everywhere. This is more general than the piecewise C^1 condition in Assumption 2.1 3-(a). In our case we cannot relax the piecewise C^1 assumption further; in Example 4 we prove that for a certain speed function $c(x,y)$ the maximizing macroscopic path actually follows the discontinuity curve of $c(x,y)$ on a set of positive measure and the set of $\mathcal{H}$ contains only piecewise C^1 paths.*

We expect that under Assumptions 2.1 and 2.2 $\Gamma_c(x,y)$ is in fact a maximum and not a supremum.

2.1. Applications to two analysable models

Variational formula (2.2) can be difficult to solve, even in simple cases. However, it does give the equivalent macroscopic last passage time model, so understanding properties of Γ_c and the optimising curves of (2.2) can be useful in developing intuition for the microscopic models.

Moreover, in special cases where we can explicitly find Γ_c we can check for properties like points of non-differentiability, points of discontinuities, concavity-breaking and flat edges, all of which are properties that are known in first passage percolation models, but do not appear in the exactly solvable models of last passage percolation.

In the (longer) Arxiv version of this paper, we analyse Γ_c and its maximising paths for two models and we encourage the interested reader to check the first version of the paper there for the details and estimates. The first example is the *shifted two-phase model* with speed function

$$c_\lambda(x,y) = \begin{cases} 1, & \text{if } y > x-\lambda, \\ r, & \text{if } y \leq x-\lambda. \end{cases} \tag{2.9}$$

We assume that $r < 1$ and $\lambda > 0$ here. Since the speed function only takes two values, the set of optimal macroscopic paths from the origin to (x, y) are piecewise linear paths. From this example we can theoretically verify that the shape Γ_{c_λ} can exhibit flat edges (irrespective of the continuous distribution of the environment), it does not need to be concave or differentiable and maximisers of the variational formula need not be unique.

The second model is the *corner-inhomogeneous* model with speed function

$$c_f(x,y) = \begin{cases} 1, & f(x) > y, \\ r, & f(x) < y, \\ 1 \wedge r, & f(x) = y. \end{cases} \tag{2.10}$$

The function f above we consider it to be a C^2 convex decreasing function $f : [0,a_0] \longrightarrow [0,b_0]$ where $f(0) = b_0 > 0$ and $f(a_0) = 0$. In words, after a bounded region of rate 1 delineated by f and the coordinate axes, the rate becomes r. Computing analytically the shape function $\Gamma_{c_f}(x,y)$ is challenging; it depends on properties of the function f. When f takes the specific form

$$f(x) = (1-\sqrt{x})^2, \quad x \in [0,1],$$

we can explicitly identify the shape function and the macroscopic maximisers of (2.2) are straight paths from $(0,0)$ to (x,y), despite the discontinuity.

For an arbitrary f, properties of macroscopic maximisers can be obtained, but not a closed form of the limiting passage time. From the fact that $c(x,y)$ is piecewise constant, macroscopic maximisers of (2.2) exist and are piecewise linear segments, one in each of the two constant regions. Depending on the shape of f they exhibit different behaviours such as following the coordinate axes. This is of particular interest as we expect that each macroscopic maximiser corresponds to microscopic maximal paths, which will stay near the coordinate axes and may alter the order of the variance of last passage times as well as the shape of the geodesic tree. The microscopic behaviour of geodesics is ongoing work.

We showcase the above theoretical results by performing some Monte Carlo simulations to show the maximal paths in different cases. Readers interested in their proofs should look at the Arxiv version 1 of this article. For all simulations we considered the curve $y = f(x)$ to be

$$f(x) = (c - x^{b/k})^k,$$

and we varied the parameters b, c, k with $b < k$ (see Fig. 1).

3. Continuity properties of $\Gamma(x, y)$

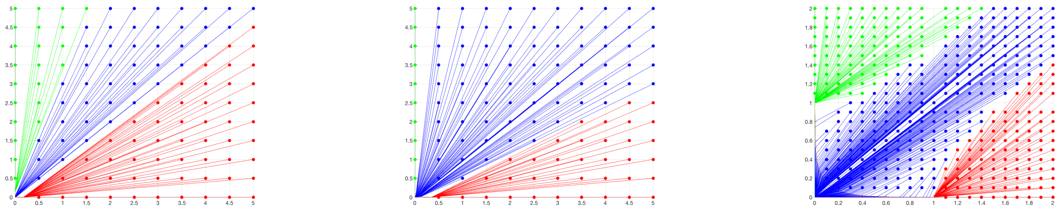
Now, we want to study what happen to the difference of the macroscopic last passage time of two points that are very close to each other.

Lemma 3.1. *Fix $a, b, z, w > 0$ and a speed function c. Then there exists a constant $C = C(a, b, z, w, c(\cdot,\cdot)) < \infty$ such that for any $\delta > 0$ we can find sufficiently small $\delta_1, \delta_2 > 0$ so that the following two regularity conditions hold: For $0 \leq a \leq z$,*

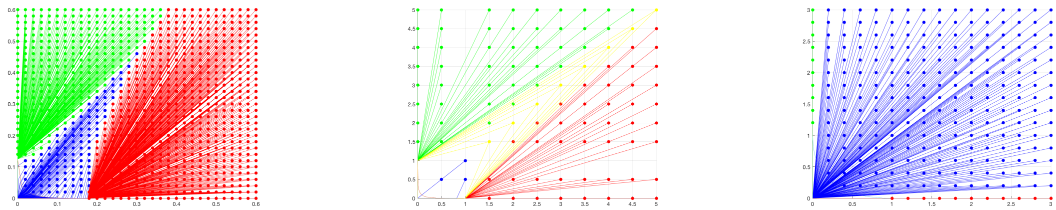
$$\Gamma((a,0),(z+\delta_1,\delta_2)) - \Gamma((a,0),(z,0)) \leq C\sqrt{\delta}. \tag{3.1}$$

For $0 \leq b \leq w$,

$$\Gamma((0,b),(\delta_1,w+\delta_2)) - \Gamma((0,b),(0,w)) \leq C\sqrt{\delta}. \tag{3.2}$$



(a) $c = 0.5, b = 1.2, k = 3, r = 3.$ (b) $c = 0.5, b = 2, k = 3, r = 3.$ (c) $c = 1, b = 1, k = 3, r = 3.$



(d) $c = 0.5, b = 1.2, k = 3, r = 4.$ (e) $c = 1, b = 1, k = 3.5, r = 3.$ (f) $c = 1, b = 1, k = 2, r = 3.$

Fig 1: (Colour online) Blue paths are maximisers which cross into the r-region from the interior of $f(x) = (c - x^{b/k})^k$. The set of all (x, y) reached by such paths may be bounded (e.g. see subfigures (D), (E)). Green and red paths are maximisers that follow either the y- or the x- axis respectively. Finally, the target points of yellow paths are those for which the maximiser is not unique.

Proof. The arguments will be symmetric, so we will prove only (3.2). We first assume that the speed function $c(x, y)$ is piecewise constant, and the different constant values are separated by the discontinuity curves. Pick a δ positive. First select $\delta_1 \in [0, 1)$, $\delta_2 \in [0, 1)$ small enough such that

1. Any discontinuity curve h_i in $[0, \delta_1] \times [0, w + \delta_2]$ is monotone and their domain is the interval $[0, \delta_1]$.
2. The intersection points of the discontinuity curves in $[0, \delta_1] \times [0, w + \delta_2]$ (if any) all lie on the y-axis.
3. It is possible for segments $[y_1, y_2]$ of the y-axis are also discontinuity curves, as long as $c(0, y) < c(x, y)$ for all $x > 0$ and $y \in (y_1, y_2)$. The x-axis does not have discontinuity segments.

The first one is possible since the h_i are finitely many in any compact set, and piecewise monotone functions. The second one because there are only finitely many intersections points.

First assume that segments of the y-axis are not discontinuity curves. Let H be the number of discontinuity curves in this rectangle (also including the north boundary in this count), and enumerate them from the lowest to the highest, including the north and south straight boundaries. Decrease δ_1 further so that

$$\max_{1 \le i \le H} \{\omega_{h_i}(\delta_1)\} < \delta$$

and select an $\eta = \eta(\delta_1) > 0$ which satisfies the condition

$$\eta \le \min_{1 \le i \le H} \{\omega_{h_i}(\delta_1)\}.$$

Keep in mind that $\eta \to 0$ as $\delta_1 \to 0$. Decrease δ_1 further so that $H\eta \ll w$. Since $c(x, y)$ is piecewise constant, we have that in-between these discontinuity curves the rates are fixed, and on the discontinuity curve the value is the smallest of the rates in the two adjacent regions by condition (1) in Assumption 2.2.

From the hypotheses so far, we have that the rectangles $Q_i = [0, \delta_1] \times [h_i(0) \wedge h_i(\delta_1), h_i(0) \vee h_i(\delta_1)]$, have completely disjoint interiors for all $1 \le i \le H$ and $c(x, y)$ takes two values. In the rectangles $R_i = [0, \delta_1] \times [h_i(0) \vee h_i(\delta_1), h_{i+1}(0) \wedge h_{i+1}(\delta_1)]$, the speed function is constant. We allow the rectangles R_i, Q_i to be degenerate horizontal lines.

For any $\mathbf{x} = (x^1(s), x^2(s)) \in \mathcal{H}(\delta_1, w + \delta_2)$ set

$$I(\mathbf{x}) = \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x^1(s), x^2(s))} ds. \tag{3.3}$$

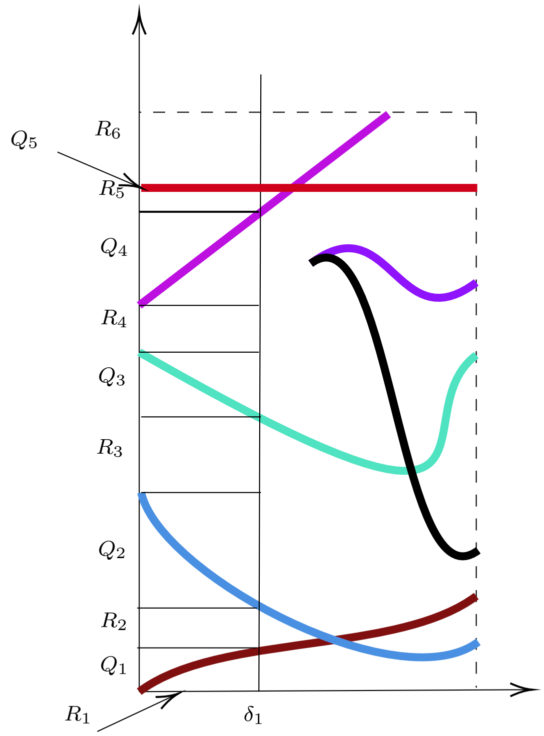


Fig 2: The selection of δ_1 and the alternating partition into R_i and Q_i rectangles needed in the proof of Lemma 3.1. In the R_i rectangles the speed function is continuous, while in the Q_i's there are monotone discontinuity curves that intersect the y-axis. δ_1 is decreased so that the discontinuity curves on $[0,\delta_1]\times[0,w+\delta_2]$ are monotone and that any discontinuities far from the y-axis (e.g. black and purple) are outside this rectangle. Moreover, R_1, and Q_5 are horizontal lines, considered trivialised rectangles.

Let $\varepsilon>0$ and assume that $\phi=(\phi^1,\phi^2)\in\mathcal{H}(\delta_1,w+\delta_2)$ is a path such that $\Gamma(\delta_1,w+\delta_2)-I(\phi)<\varepsilon$. It is possible to decompose ϕ into disjoint segments ϕ_j so that $\phi=\sum_{j=1}^{2H}\phi_j$ and that

1. For j even, $\phi_j\subseteq R_{j/2}$, and therefore it is a linear segment with derivative ϕ_j' in $\mathbb{R}^2_+$
2. For j odd, $\phi_j\subseteq Q_{(j+1)/2}$.

The sum $\sum_{j=1}^{2H}\phi_j$ means path concatenation.

For j odd, the total contribution of ϕ_j to $I(\phi)$ can be bounded by $\frac{1}{r_\ell}\gamma(\delta_1,\eta(\delta_1))$ where $r_\ell=\min\limits_{(x,y)\in[0,\delta_1]\times[0,w+\delta_2]}c(x,y)$. Over all, the total contribution of the odd-indexed segments is bounded above by $4Hr_\ell^{-1}(\eta(\delta_1)\vee\delta_1)$.

For j even, the path segment is linear and the maximum contribution of any such segment is given by

$$\begin{aligned}I(\phi_j)&=\frac{1}{r_{R_j}}\gamma(\delta_1,\text{height}(R_j))=\frac{1}{r_{R_j}}(\delta_1+\text{height}(R_j)+2\sqrt{\delta_1\text{height}(R_j)})\\&\le\frac{1}{r_{R_j}}\text{height}(R_j)+2C_j\sqrt{\delta_1}.\end{aligned}$$

Overall, on the even-indexed segments, the total contribution to $I(\phi)$ is bounded above by $\sum_{k=1}^{H}(\frac{1}{r_{R_{2k}}}\text{height}(R_{2k})+2C_{2k})\sqrt{\delta_1}\le\sum_{k=1}^{H}\frac{1}{r_{R_{2k}}}\text{height}(R_{2k})+C\sqrt{\delta_1}$.

Then,

$$\begin{aligned}\Gamma(\delta_1,w+\delta_2)-\varepsilon\le I(\phi)&\le\sum_{k=1}^{H}\frac{1}{r_{R_{2k}}}\text{height}(R_{2k})+C\sqrt{\delta_1}+4Hr_\ell^{-1}(\eta(\delta_1)\vee\delta_1)\\&\le\Gamma(0,w+\delta_2)+C\sqrt{\delta_1}+4Hr_\ell^{-1}(\eta(\delta_1)\vee\delta_1)\end{aligned}$$

$$\le \Gamma(0,w) + \frac{1}{r_\ell}\delta_2 + C\sqrt{\delta_1} + 4Hr_\ell^{-1}(\eta(\delta_1)\vee\delta_1)$$
$$\le \Gamma(0,w) + C\delta_2 \vee \sqrt{\delta_1} \vee \eta(\delta_1).$$

Finally we reach the conclusion by letting $\varepsilon \to 0$.

At this point we have verified the statement of the lemma for speed functions that are piecewise constants in-between discontinuity curves and the y-axis has no discontinuity segments.

To finish the proof in this case, consider a generic admissible speed function $c(x,y)$ and a given $\delta > 0$. Fix a tolerance $\epsilon < \delta$. Keeping the discontinuity curves from $c(x,y)$, and by possibly adding only extra vertical and horizontal discontinuity lines, we can find two piecewise constant speed functions $c_{\text{low}}(x,y)$ and $c_{\text{high}}(x,y)$ such that $c_{\text{low}}(x,y) \le c(x,y) \le c_{\text{high}}(x,y)$ such that

$$\sup_{(x,y)\in[0,1]\times[0,w+1]} \max\left\{\frac{1}{c(x,y)} - \frac{1}{c_{\text{high}}(x,y)}, \frac{1}{c_{\text{low}}(x,y)} - \frac{1}{c(x,y)}\right\} \le \epsilon.$$

Let δ_1, δ_2 be such that the lemma holds when the speed function is c_{low} (for the same δ) and decrease δ_1 even further so that in $[0,\delta_1]\times[0,w+\delta_2]$ we see no *new vertical* discontinuity lines. I.e. in $[0,\delta_1]\times[0,w+\delta_2]$ we only have the discontinuity curves of $c(x,y)$ and only new horizontal discontinuity lines for the approximation c_{low}. Then,

$$\begin{aligned}
\Gamma_c(\delta_1, w+\delta_2) - \Gamma_c(0,w) &\le \Gamma_{c_{\text{low}}}(\delta_1, w+\delta_2) - \Gamma_{c_{\text{high}}}(0,w)\\
&\le \Gamma_{c_{\text{low}}}(\delta_1, w+\delta_2) - \Gamma_{c_{\text{low}}}(0,w) + \Gamma_{c_{\text{low}}}(0,w) - \Gamma_{c_{\text{high}}}(0,w)\\
&\le C\sqrt{\delta} + 2w\epsilon \le C\sqrt{\delta},
\end{aligned}$$

which is precisely the estimate we were after.

Now suppose there are discontinuity segments on the y-axis and keep in mind that y-axis rates are lower on the segments than the interior of the rectangles they border. Partition into rectangles as before, without worrying about the y-axis, and then partitioning further we may now end-up with rectangles in which

1. R'_{2k+1} where $c(x,y)$ is constant in the interior and with a different, lower rate on the west boundary.
2. Q'_{2k} where $c(x,y)$ has a discontinuity connecting two opposite corners and a discontinuity on the west boundary.

In both of these cases, replace the rate of the region connected to the west boundary to the lower one of the boundary. With these rate values for $c(x,y)$ there are no more discontinuities on the y-axis. We only wanted to bound from above, therefore the proof is now reduced to the one before as we are only bounding from above. □

Corollary 3.2. *Fix $(x,y)\in\mathbb{R}_+^2$ and a speed function c. Then there exists $C = C(x,y,c(\cdot,\cdot)) < \infty$ such that for any δ positive, there exist δ_1, δ_2 sufficiently small*

$$\Gamma(x+\delta_1, y+\delta_2) - \Gamma(x,y) < C\delta. \tag{3.4}$$

Proof. Let $B_{(x,y)}$ be a rectangle, where the north-east corner point is (x,y) and south-west corner is $(0,0)$.
Let $\varepsilon > 0$ and ϕ^ε a path such that $\Gamma(x+\delta_1, y+\delta_2) - I(\phi^\varepsilon) < \varepsilon$. Moreover, let $\mathbf{u}$ be the point where ϕ^ε first intersects the north or the east boundary of $B_{(x,y)}$. Without loss of generality assume is the east boundary and so $\mathbf{u} = (x,b)$ for some $b\in[0,y]$. Then,

$$\begin{aligned}
\Gamma(x+\delta_1, y+\delta_2) - \varepsilon &\le I(\phi^\varepsilon)\\
&\le \Gamma(x,b) + \Gamma((x,b),(x+\delta_1,y+\delta_2))\\
&= \Gamma(x,b) + \Gamma((x,b),(x+\delta_1,y+\delta_2)) \pm \Gamma((x,b),(x,y))\\
&\le \Gamma(x,y) + \Gamma((x,b),(x+\delta_1,y+\delta_2)) - \Gamma((x,b),(x,y)).
\end{aligned}$$

A rearrangement of terms gives

$$\begin{aligned}
\Gamma(x+\delta_1, y+\delta_2) - \Gamma(x,y) &\le \Gamma((x,b),(x+\delta_1,y+\delta_2)) - \Gamma((x,b),(x,y)) + \varepsilon\\
&\le C\delta + \varepsilon
\end{aligned}$$

where we used (3.2), albeit with a starting point of (x,b). Let $\varepsilon \to 0$ to prove the corollary. □

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. Fix an $\varepsilon > 0$ and let ζ_1, ζ_2 small enough so that by Corollary 3.2 we have

$$\Gamma((a,b),(x+\zeta_3, y+\zeta_4)) - \Gamma((a,b),(x,y)) < \varepsilon/4.$$

Then, keep ζ_3, ζ_4 fixed and find a ζ_1, ζ_2 small enough so that again by Corollary 3.2,

$$\Gamma((a-\zeta_1, b-\zeta_2),(x+\zeta_3, y+\zeta_4)) - \Gamma((a,b),(x+\zeta_3, y+\zeta_4)) < \varepsilon/4.$$

Together the inequalities above give

$$\Gamma((a-\zeta_1, b-\zeta_2),(x+\zeta_3, y+\zeta_4)) - \Gamma((a,b),(x,y)) < \varepsilon/2. \tag{3.5}$$

This proves the second part of the theorem, and the outside approximation of the first part. For approximation from the inside of the rectangle, assume that on the boundary of $R((a,b)(x,y))$ no discontinuity segments exist. Then Lemma 3.1 can be iteratively applied and we can find positive ζ_5, ζ_6, ζ_7, ζ_8 so that

$$\Gamma((a,b),(x,y)) - \Gamma((a+\zeta_5, b+\zeta_6),(x-\zeta_7, y-\zeta_8)) < \varepsilon/2. \tag{3.6}$$

Let $\delta_0 = \min_{1\le i\le 8}\{\zeta_i\}$. Since $\Gamma(u,v)$ decreases in the first argument and increases in the second argument the inequalities (3.5) and (3.6), together with our choice of δ_0 give

$$\Gamma((a-\delta_0, b-\delta_0),(x+\delta_0, y+\delta_0)) - \Gamma((a+\delta_0, b+\delta_0),(x-\delta_0, y-\delta_0)) < \varepsilon.$$

and that for any $\tilde{a} \in [a-\delta_0, a+\delta_0]$, $\tilde{b} \in [b-\delta_0, b+\delta_0]$, $\tilde{x} \in [x-\delta_0, x+\delta_0]$, $\tilde{y} \in [y-\delta_0, y+\delta_0]$, we have

$$\Gamma((a+\delta_0, b+\delta_0),(x-\delta_0, y-\delta_0)) \le \Gamma((\tilde{a},\tilde{b}),(\tilde{x},\tilde{y})) \le \Gamma((a-\delta_0, b-\delta_0),(x+\delta_0, y+\delta_0)).$$

The last two inequalities combined give the first part of the theorem. $\square$

The reason for this technical approximation is the statements in the next lemma, motivated by the following argument. In the simplest case we would like to approximate the limits of last passage times using the limiting Γ_c in rectangles where $c(x,y)$ has one discontinuity line. Unfortunately, unless the discontinuity of the speed is a line of slope 1, we cannot say at this point that the limit is $\Gamma_c(x,y)$. However, if the speed function is continuous, the fact that the limit of passage times is Γ_c in that environment is given by Theorem 3.1. in [18]. So we may approximate Γ_c with the value $\Gamma_{\tilde{c}}$ where $\tilde{c}$ will be a continuous speed function that approximates $c(s,t)$.

Lemma 3.3 (Continuity of Γ in the speed function)**.** *Let $c(s,t)$ take only two values r_1, r_2 in two regions of $[a,x]\times[b,y]$ separated by a weakly monotone curve h, which satisfies Assumption 2.1. Then, for every $\varepsilon > 0$ there exists a $\eta_{h,\varepsilon} > 0$ so that for all $\eta < \eta_{h,\varepsilon}$ there exists a continuous speed function $c^{cont}_{\eta}(s,t) \le c(s,t)$ so that*

$$\Gamma_{c^{cont}_{\eta}}((a,b)(x,y)) - \Gamma_c((a,b),(x,y)) \le \varepsilon.$$

Proof of Lemma 3.3. Fix (x,y) and without loss assume that the starting point is $(a,b) = (\alpha, 0)$ for some $\alpha > 0$. We present the case when the curve h starts somewhere on $[\alpha, x]$ and exits somewhere on the east boundary $\{x\}\times[0,y]$ and the rates above the curve is $r_1 < r_2$. Symmetric arguments as the one below will work in all other cases, and are left to the reader.

For a fixed $\varepsilon > 0$ we can find an $\eta_{\varepsilon,h} > 0$ so that for all postive $\eta < \eta_{\varepsilon,h}$ we have $|\Gamma_c((\alpha-\eta,0),(x-\eta,y)) - \Gamma_c((\alpha,0),(x,y))| < \varepsilon$. This is possible by Theorem 2.4. Fix any such η and define the curve h_η by the relation $h_\eta(t) = h(t+\eta)$, i.e. this correspond to shift of h by η to the right. Then, we define a speed function $c_\eta(\cdot,\cdot)$ on $[\alpha,x]\times[0,y]$

$$c_\eta(z,w) = \begin{cases} r_1, & \text{if } (z,w) \text{ is above or on the graph of } h_\eta, \\ r_2, & \text{otherwise.} \end{cases}$$

We make two observations:

1. $c(z,w) \ge c_\eta(z,w)$ for all $(z,w) \in [\alpha,x]\times[0,y]$, giving $\Gamma_{c_\eta}((\alpha,0),(x,y)) \ge \Gamma_c((\alpha,0),(x,y))$.

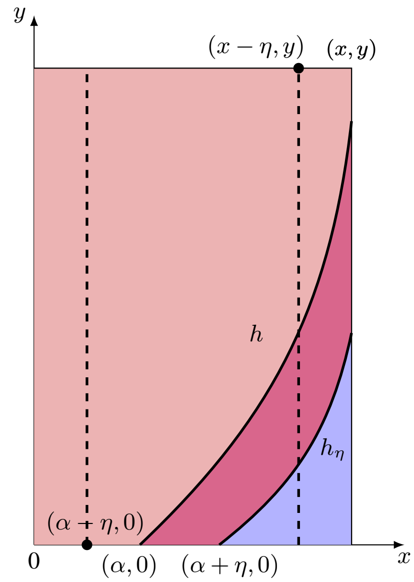


Fig 3: Graphical representation for the proof of Lemma 3.3.

2. By construction

$$\Gamma_c((\alpha-\eta,0),(x-\eta,y)) = \Gamma_{c_\eta}((\alpha,0),(x,y)). \tag{3.7}$$

From these observations we define a new, continuous function $c_\eta^{\text{cont}}(\cdot,\cdot)$ on $[\alpha,x]\times[0,y]$ so that

$$c_\eta(z,w) \le c_\eta^{\text{cont}}(z,w) \le c(z,w), \quad \text{for all } (z,w)\in[\alpha,x]\times[0,y].$$

This and (3.7) imply

$$\Gamma_{c_\eta^{\text{cont}}}((\alpha,0),(x,y)) \le \Gamma_{c_\eta}((\alpha,0),(x,y)) = \Gamma_c((\alpha-\eta,0),(x-\eta,y)) \le \Gamma_c((\alpha,0),(x,y))+\varepsilon, \tag{3.8}$$

which in turn yields the Lemma. □

4. Proof of Theorem 2.6

To prove Theorem 2.6 we need a few preliminary lemmas which help us define some useful properties of the last passage time in a 2D inhomogeneous environment.

We begin by identifying the last passage time limits in simple cases of speed function, that will be used as building blocks for approximations to the general case. We first find the law of large numbers without fixing the maximal path but forcing it to stay in a homogeneous corridor. Let the speed function be

$$c(x,y) = \begin{cases} r_2 & y > x+\lambda, \\ r_1 & x-\lambda \le y \le x+\lambda, \\ r_3 & y < x-\lambda. \end{cases} \tag{4.1}$$

with $\lambda\in\mathbb{R}_+$.

Lemma 4.1 (Passage times in homogeneous corridors)**.** *Assume $c(x,y)$ in (4.1) for all $(x,y)\in(0,b)\times(0,e)$. Let $(z,w)\in(0,b]\times(0,e]$ with $w\in(z-\lambda,z+\lambda)$ and let $\tilde G_{\lfloor nz\rfloor,\lfloor nw\rfloor}$ be the last passage time from $(0,0)$ to $(\lfloor nz\rfloor,\lfloor nw\rfloor)$ subject to the constraint that*

admissible paths stay in the r_1-rate region inside the strip $\lfloor nb\rfloor-\lambda\le\lfloor ne\rfloor\le\lfloor nb\rfloor+\lambda$, except possibly for a bounded number of initial and final steps.

Then

$$\lim_{n\to\infty} n^{-1}\tilde{G}_{\lfloor nz\rfloor,\lfloor nw\rfloor} = r_1^{-1}\gamma(z,w), \quad \mathbb{P}-a.s. \tag{4.2}$$

Proof. To obtain the upper bound $\lim_{n\to\infty} n^{-1}\tilde{G}_{(\lfloor nz\rfloor,\lfloor nw\rfloor)} \le r_1^{-1}\gamma(z,w)$ ignore the path restrictions and assume that the environment in the whole region is homogeneous of constant rates r_1.

For the lower bound we use a coarse graining argument, taking into account the path restrictions. Fix an $\varepsilon\in(0,1)$ and consider the points

$$\mathscr{P}_{z,w,\varepsilon} = \{(k\lfloor\varepsilon nz\rfloor, k\lfloor\varepsilon nw\rfloor) : k=1,2,\dots,\lfloor\varepsilon^{-1}\rfloor\}\cup\{\lfloor nz\rfloor,\lfloor nw\rfloor)\}.$$

To bound $\tilde{G}_{\lfloor nz\rfloor,\lfloor nw\rfloor}$ from below, force the path to go through the partition points of $\mathscr{P}_{z,w,\varepsilon}$. By possibly reducing ε further, for each $1\le k\le\lfloor\varepsilon^{-1}\rfloor$, each rectangle with lower-left and upper-right corners two consecutive points of $\mathscr{P}_{z,w,\varepsilon}$ is completely inside the region of rate r_1. For these rectangles we allow the path segments to explore space.

For $2\le k<\lfloor\varepsilon^{-1}\rfloor$ let $G_{R_k^n}$ be the last passage time from $((k-1)\lfloor\varepsilon nz\rfloor,(k-1)\lfloor\varepsilon nw\rfloor)$ to $(k\lfloor nz\varepsilon\rfloor, k\lfloor nw\varepsilon\rfloor)$. R_k^n refers to the rectangle that contains all the admissible paths between the two points.

Let $0\le\delta=\delta(\varepsilon)<\varepsilon r^{-1}\gamma(z,w)$ and assume without loss that $\delta/\varepsilon\to 0$ as $\varepsilon\to 0$. A large deviation estimate (Theorem 4.1 in [34]) gives a constant $C=C(r,z,w,\varepsilon,\delta)$ such that for k fixed

$$\mathbb{P}\{G_{R_k^n}\le n(\varepsilon r^{-1}\gamma(z,w)-\delta)\}\le e^{-Cn^2}. \tag{4.3}$$

The sequence of passage times $\{G_{R_k^n}\}_k$ are i.i.d. and as such, a Cramèr large deviation estimate and a Borel-Cantelli argument give for large n,

$$\tilde{G}_{\lfloor nz\rfloor,\lfloor nw\rfloor}\ge\sum_{k=1}^{\lfloor\varepsilon^{-1}\rfloor-1} G_{R_k^n}\ge n(\lfloor\varepsilon^{-1}\rfloor-1)(\varepsilon r^{-1}\gamma(z,w)-\delta), \quad \mathbb{P}\text{-a.s.}$$

Divide the inequality through by n and take the $\liminf$ as $n\to\infty$. After that, send $\varepsilon\to 0$ to finish the proof. □

From the coarse graining argument in the previous proof, we see that when we restrict to maximal paths in a narrow (but macroscopic) homogeneous corridor we still obtain the same limiting passage time as if the environment was homogeneous throughout. This is a consequence of the microscopic fluctuations of the maximal paths and the strict concavity of γ. As the width ε of the corridor tends to 0, the limiting shape of the corridor is a straight line, which is the shape of the macroscopic maximal path in a homogeneous region.

Lemma 4.2 (Passage times in C^1 homogeneous corridors)**.** *Let $\mathbf{x}(\mathbf{s})$ be a C^1 increasing path from (a,b) to (c,d), and let $\mathcal{N}(\mathbf{x},\varepsilon)$ be a neighborhood subject to the constraint that $c(\mathbf{x}(\mathbf{s}))=r$ (constant) on $\mathcal{N}(\mathbf{x},\varepsilon)$. Let $G^{(n)}_{n\mathcal{N}(\mathbf{x},\varepsilon)}$ be the passage time from $\lfloor n(a,b)\rfloor$ to $\lfloor n(c,d)\rfloor$, subject to the constraint that maximal paths never exit $n\mathcal{N}(\mathbf{x},\varepsilon)$. Then*

$$\varliminf_{n\to\infty} n^{-1}G^{(n)}_{n\mathcal{N}(\mathbf{x},\varepsilon)}\ge\frac{1}{r}\int_0^1\gamma(\mathbf{x}'(s))\,ds.$$

Proof. Consider a partition of the interval $[0,1]$ $\mathcal{P}=\{0=s_0<s_1<\ldots<s_N=1\}$ fine enough so that the rectangles $R(\mathbf{x}(s_i),\mathbf{x}(s_{i+1}))$ are completely inside the neighborhood $\mathcal{N}(\mathbf{x},\varepsilon)$. Then,

$$\begin{aligned}
\varliminf_{n\to\infty} n^{-1}G^{(n)}_{n\mathcal{N}(\mathbf{x},\varepsilon)} &\ge \varliminf_{n\to\infty} n^{-1}\sum_{i=0}^{N-1} G^{(n)}_{\lfloor n\mathbf{x}(s_i)\rfloor,\lfloor n\mathbf{x}(s_{i+1})\rfloor}\ge\sum_{i=0}^{N-1}\varliminf_{n\to\infty} n^{-1}G^{(n)}_{\lfloor n\mathbf{x}(s_i)\rfloor,\lfloor n\mathbf{x}(s_{i+1})\rfloor}\\
&\ge\frac{1}{r}\sum_{i=0}^{N-1}\gamma(\mathbf{x}(s_{i+1})-\mathbf{x}(s_i))=\frac{1}{r}\sum_{i=0}^{N-1}\gamma\Big(\frac{\mathbf{x}(s_{i+1})-\mathbf{x}(s_i)}{s_{i+1}-s_1}\Big)(s_{i+1}-s_i)\\
&=\frac{1}{r}\sum_{i=0}^{N-1}\gamma\big(\mathbf{x}'(\xi_i)\big)(s_{i+1}-s_i),\ \text{for some } \xi_i\in[s_i,s_{i+1}], \text{by the mean value theorem.}
\end{aligned}$$

As the mesh of the partition tends to 0, the last line converges to $\frac{1}{r}\int_0^1 \gamma(\mathbf{x}'(s))\,ds$, as it is a Riemann sum. This gives the result. □

Lemma 4.3 (Passage times in two-phase rectangles)**.** *Consider a C^1 function $h : [0,a] \to [0,b]$ and a macroscopic rectangle $[0,a]\times[0,b]$ and in which the speed function is*

$$c(x,y) = r_1 \mathbb{1}_{\{y>h(x)\}} + r_2 \mathbb{1}_{\{y<h(x)\}} + r_1 \wedge r_2 \mathbb{1}_{\{y=h(x)\}}.$$

We further assume that

1. *$h([0,a]) = [0,b]$, h is monotone and $h(x) \notin \{0,b\}$, for any $x \in (0,a)$.*
2. *There exists $\eta > 0$ so that $\min_{x\in(0,a)} |h'(x)| > \eta > 0$.*
3. *If h is increasing, then we further assume that for the same $\eta > 0$ as in (2), we have $\sup_{x\in(0,a)} \left| h'(x) - \frac{b}{a} \right| < \eta$. In particular, the first derivative is bounded and there exists a constant L so that the curve is Lipschitz-L.*

Assume for convenience that $r_1 < r_2$. Then, there exists a uniform constant C_h so that last passage time limits satisfy

1. *For h increasing ,*

$$\frac{1}{r_1}\gamma(a,b) - \frac{2}{r_1}C_h \mathrm{length}(h)\eta \le \varliminf_n n^{-1} G^{(n)}_{\lceil na\rceil,\lceil nb\rceil} \le \varlimsup_n n^{-1} G^{(n)}_{\lceil na\rceil,\lceil nb\rceil} \le \frac{1}{r_1}\gamma(a,b). \tag{4.4}$$

   Moreover,

$$\frac{1}{r_1}\gamma(a,b) - \frac{2}{r_1}C_h \mathrm{length}(h)\eta \le \Gamma(a,b) < \frac{1}{r_1}\gamma(a,b), \tag{4.5}$$

   which in turn implies

$$\varlimsup_{n\to\infty} |n^{-1} G^{(n)}_{\lceil na\rceil,\lceil nb\rceil} - \Gamma(a,b)| \le \frac{2}{r_1} C_h \mathrm{length}(h)\eta. \tag{4.6}$$

2. *When h is decreasing*

$$\lim_{n\to\infty} n^{-1} G^{(n)}_{\lceil na\rceil,\lceil nb\rceil} = \Gamma(a,b). \tag{4.7}$$

Proof. We first treat the case of increasing h. Without loss, assume $h(0) = 0$ and $h(a) = b$. Since $r_1 < r_2$ we obtain the upper bound in (4.4) if we lower r_2 to r_1 and assume a homogeneous environment with constant speed function $c_{\mathrm{low}}(x,y) = r_1$. This also gives the upper bound in (4.5) since $c_{\mathrm{low}}(x,y) \le c(x,y)$.

Now for the lower bound. Let $\varepsilon > 0, \delta > 0$ sufficiently small. First consider a graph $h_\varepsilon(x) = (h(x)+\varepsilon)\wedge b$ which lies solely in the r_1 region of $c(x,y)$.

By hypothesis (1), assume ε is small enough so that the first time h_ε touches the top boundary $[0,a]\times\{b\}$, is precisely at some point $x_\varepsilon > a-\delta$. Consider a parametrisation for h, $(h^{(1)}(s), h^{(2)}(s)) : [0,1] \to \mathbb{R}^2$. Then point x_ε corresponds to some $1 - s_\varepsilon \in [0,1]$.

Then define the curve $\mathbf{x}$ that goes from $(0,0)$ to $(0, h_\varepsilon(0))$ by time s_ε, then follows h_ε until it takes the value b by time 1 and then stays on the north boundary at value b for time s_ε.

Since h is rectifiable, so is h_ε, and we assume without loss that h_ε has the Lipschitz parametrization

$$\Big(h^{(1)}\Big((s-s_\varepsilon)\frac{1-s_\varepsilon}{1-2s_\varepsilon}\Big), h^{(2)}\Big((s-s_\varepsilon)\frac{1-s_\varepsilon}{1-2s_\varepsilon}\Big)+\varepsilon\Big), \quad s \in [s_\varepsilon, 1-s_\varepsilon].$$

Then we estimate

$$\begin{aligned}
\Gamma(a,b) &\ge \int_{s_\varepsilon}^{1-s_\varepsilon} \frac{\gamma(\mathbf{x}'(s))}{r_1}\,ds = \frac{1-s_\varepsilon}{1-2s_\varepsilon}\int_0^{1-s_\varepsilon} \frac{\gamma(h^{(1)'}(s), h^{(2)'}(s))}{r_1}\,ds \\
&= \frac{1-s_\varepsilon}{1-2s_\varepsilon}\int_0^{1-s_\varepsilon} h^{(1)'}(s)\frac{\gamma(1, \frac{h^{(2)'}(s)}{h^{(1)'}(s)})}{r_1}\,ds = \frac{1-s_\varepsilon}{1-2s_\varepsilon}\int_0^{1-s_\varepsilon} h^{(1)'}(s)\frac{\gamma(1, h'(h^{(1)}(s)))}{r_1}\,ds \\
&= \frac{1-s_\varepsilon}{1-2s_\varepsilon}\int_0^{h^{(1)}(1-s_\varepsilon)} \frac{\gamma(1,h'(u))}{r_1}\,du \ge \frac{1-s_\varepsilon}{1-2s_\varepsilon}\int_0^{h^{(1)}(1-s_\varepsilon)} \frac{\gamma(1,\frac{b}{a}-\eta)}{r_1}\,du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-s_\varepsilon}{1-2s_\varepsilon} h^{(1)}(1-s_\varepsilon)\frac{\gamma(1,\frac{b}{a}-\eta)}{r_1} \\
&\geq a\frac{\gamma(1,\frac{b}{a}-\eta)}{r_1} - \delta\frac{1-s_\varepsilon}{1-2s_\varepsilon}\frac{\gamma(1,\frac{b}{a}-\eta)}{r_1} - \frac{s_\varepsilon}{1-2s_\varepsilon}\frac{\gamma(1,\frac{b}{a}-\eta)}{r_1}. \qquad (4.8)
\end{aligned}
$$

Letting $\varepsilon \to 0$ makes the last term vanish, and by then letting $\delta \to 0$ we obtain

$$
\Gamma(a,b) \geq \frac{\gamma(a, b-a\eta)}{r_1} = \frac{1}{r_1}\Big(a+b-a\eta+2\sqrt{a}\sqrt{b}\sqrt{1-\frac{a\eta}{b}}\Big). \qquad (4.9)
$$

By the mean value theorem $\eta < \min|h'(s)| < ba^{-1}$ and by item (2) in the hypothesis, one can check that

$$
\sqrt{1-\frac{a\eta}{b}} \geq 1-\frac{a\eta}{b}.
$$

We now estimate the γ-term in the left hand side of (4.9).

$$
\begin{aligned}
\gamma(a,b-a\eta) &= a+b-a\eta+2\sqrt{a}\sqrt{b}\Big(1-\frac{a\eta}{b}\Big) = a+b-a\eta+2\sqrt{a}\sqrt{b}-2\frac{a^{3/2}\eta}{b^{1/2}} \qquad (4.10)\\
&\geq \gamma(a,b) - 2\eta\Big(a+\frac{a^{3/2}}{b^{1/2}}\Big). \qquad (4.11)
\end{aligned}
$$

Now the lower bound in (4.5). Let

$$
C_h^2 > \frac{1+2\sqrt{L}}{L^3} \vee \Big(1+\frac{1+2\sqrt{L}}{\min_{x\in(0,a)} h'(x)}\Big).
$$

Keep in mind that by the mean value theorem, $b/a \geq \min_{x\in(0,a)} h'(x)$ and by the choice of C_h we have

$$
\frac{b}{a} \geq \min_{x\in(0,a)} h'(x) \geq \frac{1+2\sqrt{L}}{C_h^2-1}.
$$

Then we can bound

$$
\begin{aligned}
0 &\leq a^2((C_h^2-1)b-(1+2\sqrt{L})a) < (C_h^2-1)a^2b-(1+2\sqrt{L})a^3+C_h^2b^3\\
&= (C_h^2-1)a^2b - a^3 - 2\sqrt{L}a^3 + C_h^2b^3 < (C_h^2-1)a^2b - a^3 - 2a^{5/2}b^{1/2} + C_h^2b^3.
\end{aligned}
$$

In the last inequality above we used (3), since it implies $h(a)-h(0) = b \leq La$. An equivalent way to write the last inequality is

$$
\Big(a+\frac{a^{3/2}}{b^{1/2}}\Big)^2 < C_h^2(a^2+b^2). \qquad (4.12)
$$

From (4.12), we conclude that $a+\frac{a^{3/2}}{b^{1/2}} < C_h\sqrt{a^2+b^2} \leq C_h \mathrm{length}(h)$. Substitute this in (4.11) to finally prove the lower bound in (4.5).

For the lower bound in (4.4) consider again the function h_ε and s_ε from before and consider a partition of $[0, 1-s_\varepsilon]$, $\mathscr{P}_{s_\varepsilon,\delta} = \{x_k = k\delta(1-s_\varepsilon)\}_{0\leq k\leq \lfloor \delta^{-1}\rfloor}$, of mesh $\delta > 0$. We assume the partition is fine enough so that the rectangles $R_k = [x_k, x_{k+1}] \times [h_\varepsilon(x_k), h_\varepsilon(x_{k+1})]$ completely lie in the homogeneous region of rate r_1 and so that Riemann sum

$$
\sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} r_1^{-1}\gamma(h^{(1)'}(x_{k+1}), h^{(2)'}(x_{k+1}))(x_{k+1}-x_k) \geq \int_0^{1-s_\varepsilon} \frac{\gamma(h^{(1)'}(s), h^{(2)'}(s))}{r_1}\,ds - \theta_1 \qquad (4.13)
$$

for some fixed tolerance $\theta_1 > 0$. Moreover, assume the partition is fine enough so that for η_1 sufficiently small, with $0 < \eta_1 < \alpha$

$$
\Big|\frac{h^{(i)}(x_{k+1})-h^{(i)}(x_k)}{x_{k+1}-x_k} - h^{(i)'}(x_{k+1})\Big| < \eta_1, \quad \text{for } i = 1,2.
$$

Finally, fix a small $\theta_2 > 0$ and let n large enough so that Theorem 4.1 in [34] gives

$$\mathbb{P}\{G_{nR_k} < nr_1^{-1}\gamma(h^{(1)}(x_{k+1}) - h^{(1)}(x_k), h_\varepsilon^{(2)}(x_{k+1}) - h_\varepsilon^{(2)}(x_k)) - n\theta_2\} \leq e^{-cn}.$$

By the Borel-Cantelli lemma we can then let n be large enough so that $\mathbb{P}$-a.s. for all k

$$G_{nR_k} > nr_1^{-1}\gamma(h^{(1)}(x_{k+1}) - h^{(1)}(x_k), h_\varepsilon^{(2)}(x_{k+1}) - h_\varepsilon^{(2)}(x_k)) - n\theta_2.$$

Above we denoted by G_{nR_k} the maximum weight that can be collected from oriented paths in the set nR_k.

By superadditivity, the passage times satisfy

$$\begin{aligned}
G^{(n)}_{\lceil na\rceil,\lceil nb\rceil} &\geq \sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} G_{nR_k} \geq n\sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} r_1^{-1}\gamma(h^{(1)}(x_{k+1}) - h^{(1)}(x_k), h_\varepsilon^{(2)}(x_{k+1}) - h_\varepsilon^{(2)}(x_k)) - n\theta_2\delta^{-1}\\
&= n\sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} r_1^{-1}\gamma\Big(\frac{h^{(1)}(x_{k+1}) - h^{(1)}(x_k)}{x_{k+1}-x_k}, \frac{h_\varepsilon^{(2)}(x_{k+1}) - h_\varepsilon^{(2)}(x_k)}{x_{k+1}-x_k}\Big)(x_{k+1}-x_k) - n\theta_2\delta^{-1}\\
&\geq n\sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} r_1^{-1}\gamma(h^{(1)'}(x_{k+1}) - \eta_1, h^{(2)'}(x_{k+1}) - \eta_1)(x_{k+1}-x_k) - n\theta_2\delta^{-1}\\
&\geq n\sum_{k=0}^{\lfloor\delta\rfloor^{-1}-1} r_1^{-1}\gamma(h^{(1)'}(x_{k+1}), h^{(2)'}(x_{k+1}))(x_{k+1}-x_k) - \frac{n}{r_1}\omega_\gamma(\eta_1) - n\theta_2\delta^{-1},\\
&\geq n\int_0^{1-s_\varepsilon}\frac{\gamma(h'(s))}{r_1}\,ds - \frac{n}{r_1}\omega_\gamma(\eta_1) - n\theta_1 - n\theta_2\delta^{-1}, \text{ by (4.13)}.
\end{aligned}$$

Divide through by n and take the $\underline{\lim}$ on both sides. First let $\theta_1, \theta_2 \to 0$. After that take $\eta_1 \to 0$. The final estimate comes from a repetition of computation (4.8) and bounds (4.11), (4.12).

When h is decreasing, the approximation argument is simpler. We briefly highlight it but leave the details to the reader. First of all, any monotone curve from $[0,a]$ to $[0,b]$ will have to cross h at a unique point $(\zeta, h(\zeta))$. Then from Jensen's inequality, the piecewise linear curve from 0 to $(\zeta, h(\zeta))$ and then to (a,b) achieves a higher value for the functional (2.3). So, candidate macroscopic optimisers can be restricted to piecewise linear curves, and this gives the lower bound

$$\Gamma(a,b) \leq \varliminf_{n\to\infty} n^{-1}G^{(n)}_{\lfloor na\rfloor,\lfloor nb\rfloor}$$

by a coarse graining argument as for the case when h was increasing. For the upper bound, partition the curve h finely enough with a mesh $\delta > 0$. Any microscopic optimal path will have to cross the microscopic curve $[nh]$ at some point $(\lfloor n\zeta\rfloor, \lfloor n(h(\zeta))\rfloor)$, lying between two of the partition points. For n large enough, the passage time on this path will $\mathbb{P}$-a.s , be no more than $nr_1^{-1}\gamma(\zeta, h(\zeta)) + nr_2^{-1}\gamma(a-\zeta, b-h(\zeta)) + n\varepsilon + Cn\sqrt{\delta}$ for any fixed ε. Divide by n, take the quantifiers to 0 and then take supremum over all crossing points to obtain the upper bound. $\square$

Example. Consider a square with south-west corner $(0,0)$ and north-east corner $(1,1)$. This square is subdivided in two constant-rate regions by a parabola $h(x) = x^2$ where above the rate is 1 and below is $r \in (0,1)$. Then the set of the all potential optimisers is a concatenation of straight lines in the 1 region and convex segments along the discontinuity $h(x)$.

From Jensen's inequality and the convexity of $h(x)$ it is immediate to see that any segment of an optimiser in the rate 1 region will have to be a straight line from the entry point to the exit point of the optimiser in the region. Therefore it remains to prove the shape of the maximal path in the r region.

We first claim that for any potential optimiser $\ell \in \mathcal{H}(1,1)$, there exists a neighborhood $\mathcal{N}_\ell$ on $[0,1]$ such that for every $x \in \mathcal{N}_\ell$ a potential optimiser in $\mathcal{H}(1,1)$ takes the value $h(x)$ for $x \in \mathcal{N}_\ell$.

To see this we use a proof by contradiction: First, we show that for r small enough, any potential optimiser has to enter the r-region. If that was not the case, Jensen's inequality would give that the straight line from $(0,0)$ to $(1,1)$ is actually an optimiser and the last passage time constant would be

$$I_\ell(1,1) = \int_0^1 (\sqrt{1}+\sqrt{1})^2 dt = 4.$$

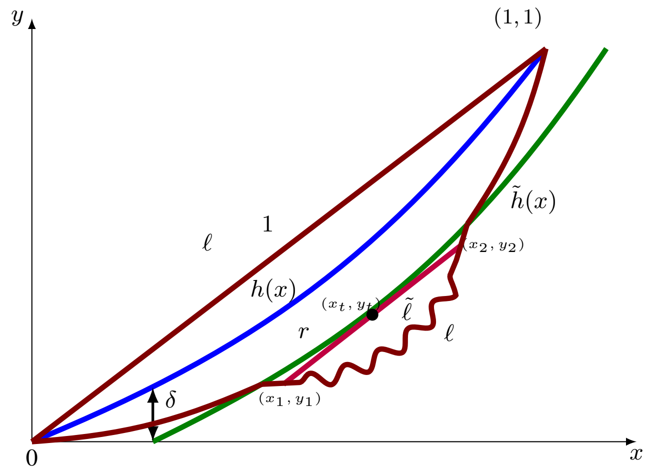


Fig 4: Graphical representation for Example 4.

However, the C^1 curve $h(x)$ is also an admissible curve, and it achieves potential

$$I_{h(x)}(1,1) = \frac{1}{r}\int_0^1 (1+\sqrt{2t})^2 dt = \frac{2}{r}\Big(1+\frac{2}{3}\sqrt{2}\Big),$$

by the lower semicontinuity assumption on $c(x, y)$. Therefore, for $r < \frac{1}{2} + \frac{\sqrt{2}}{3}$, we have $I_\ell(1,1) < I_{h(x)}(1,1)$, so the optimiser ℓ has to enter the slow region.

Now suppose that $r < \frac{1}{2} + \frac{\sqrt{2}}{3}$ in order to complete the example. We can find points $(a, h(a))$ and $(b, h(b))$ so that ℓ enters in the r region through the point $(a, h(a))$ with $a \in [0,1)$ and stays in there without touching $h(x)$ except until $(b, h(b))$. We allow that potentially $(1,1) = (b, h(b))$. Since ℓ is continuous, it is possible to find a $\delta > 0$ so that for t in some open interval $\mathcal{N}_\ell$ we have

$$|h(t) - \ell(t)| > \delta. \tag{4.14}$$

To see that (4.14) is not respected by a potential optimiser, consider a δ shift $\tilde h = (h - \delta/2)^+$. Since ℓ is continuous it will cross $\tilde h$ at least in two points $(a_1, \tilde h(a_1))$ and $(b_1, \tilde h(b_1))$ and without loss assume $[a_1, b_1] \subseteq \mathcal{N}_\ell$. Pick any $t \in (a_1, b_1)$ and consider the tangent line at $(t, \tilde h(t))$ on $\tilde h$. By construction, this should cross ℓ in (x_1, y_1) and (x_2, y_2) (see Figure 4). By Jensen's inequality we know that the path $\tilde\ell$ which goes through ℓ up to point (x_1, y_1), straight to (x_2, y_2) and then follows ℓ. Then, $I(\tilde\ell) > I(\ell)$ and therefore, ℓ cannot be an optimiser. This gives the desired contradiction.

The contradiction was reached by assuming that a potential optimiser enters the slow region, but without following the discontinuity curve h. This completes the example. □

Remark 4.4. *In the above example, we only used the explicit form of the discontinuity h just to argue that a potential optimiser will eventually enter the slow region. If this information is known, the latter part of the proof is completely general and it uses local convexity properties of the discontinuity. In particular it just uses the fact that the discontinuity curve and the potential optimiser are continuous, piecewise C^1 and there exists a point $(t, h(t))$ for which the tangent line does not enter the fast region.* □

Remark 4.5. *The previous example suggests that potential optimisers cannot be more regular than the discontinuity curves.* □

Lemma 4.6 (Exponential concentration of passage times with continuous speed)**.** *Let $c(s, t)$ be a continuous speed function in $[0, x] \times [0, y]$. Then, for any $\theta > 0$, there exists constants A and $\kappa_{\theta,c}$*

$$\mathbb{P}\{G^{(n)}_{\lceil nx\rceil, \lceil ny\rceil} \ge n\Gamma_c(x,y) + n\theta\} \le Ae^{-\kappa_{\theta,c} n}. \tag{4.15}$$

Proof of Lemma 4.6. Fix a tolerance ε small. Its size will be determined in the proof. For a $K \in \mathbb{N}$, consider the two partitions

$$\mathscr{P}^{(K)}_x = \{\alpha_\ell = \ell x K^{-1}\}_{0\le \ell\le K}, \text{ and } \mathscr{P}^{(K)}_y = \{\beta_\ell = \ell y K^{-1}\}_{0\le \ell\le K}$$

of $[0,x]$ and $[0,y]$ respectively. Let $R_{i,j}$ denote the open rectangle with south-west corner (α_i,β_j). Let

$$r_{i,j}=\inf_{(s,t)\in R_{i,j}} c(s,t).$$

Define a speed function

$$c_{\text{low}}(s,t)=\sum_{(i,j)} r_{i,j}\mathbb{1}_{\{(s,t)\in R_{i,j}\}}+\sum_{(i,j)} r_{i-1,j}\wedge r_{i,j}\mathbb{1}_{\{s=\alpha_i,\beta_j<t<\beta_{j+1}\}}+\sum_{(i,j)} r_{i,j-1}\wedge r_{i,j}\mathbb{1}_{\{\alpha_i<s<\alpha_{i+1},t=\beta_j\}}.$$

The value of $c(\alpha_i,\beta_j)$ is the minimum of the values in a neighborhood around it.

We are assuming the initial condition that $r_{i,-1}=r_{-1,j}=\infty$. In words, $c_{\text{low}}(s,t)$ is a step function with the minimum value of the neighbouring rates on the boundaries of $R_{i,j}$. Note that $c_{\text{low}}(s,t)\le c(s,t)$. Let $\overline{R}_{i,j}$ denote the rectangle together with any of its boundaries for which it contributed the rate, using some rules to break ties, if the boundary value agrees for two rectangles.

At this point we assume that $K=K(\varepsilon)$ is large enough so that $\|c-c_{\text{low}}\|_\infty<\varepsilon$. This implies that

$$\Gamma_{c_{\text{low}}}(x,y)-\Gamma_c(x,y)\le\varepsilon\gamma(x,y)r_{\min}^{-2},$$

where $r_{\min}$ is the smallest value of $c(x,y)$. This is because for any path $\mathbf{x}$,

$$\begin{aligned}
&\int_0^1\Big\{\frac{\gamma(\mathbf{x}'(s))}{c_{\text{low}}(x_1(s),x_2(s))}-\frac{\gamma(\mathbf{x}'(s))}{c(x_1(s),x_2(s))}\Big\}ds\\
&\quad=\int_0^1\frac{\gamma(\mathbf{x}'(s))(c(x_1(s),x_2(s))-c_{\text{low}}(x_1(s),x_2(s)))}{c(x_1(s),x_2(s))c_{\text{low}}(x_1(s),x_2(s))}ds\le\epsilon\int_0^1\frac{\gamma(\mathbf{x}'(s))}{c^2(x_1(s),x_2(s))}ds\\
&\quad\le\epsilon r_{\min}^{-2}\gamma(x,y),
\end{aligned}$$

and the bound extends to the supremum over paths $\mathbf{x}$.

Pick a $L>0$ so that $L^{-1}\ll K^{-1}$ and further partition each axis segment

$$\mathscr{H}_i^{(L)}=\{\alpha_i+\ell(\alpha_{i+1}-\alpha_i)L^{-1}\}_{0\le\ell\le L},\text{ and }\mathscr{V}_j^{(L)}=\{\beta_i+\ell(\beta_{i+1}-\beta_i)L^{-1}\}_{0\le\ell\le L}.$$

Define

$$\mathscr{D}_{i,j}=\{\mathbf{d}_{i,j}^\ell=(\alpha_i+\ell(\alpha_{i+1}-\alpha_i)L^{-1},\beta_j)\},\qquad\mathscr{E}_{i,j}=\{\mathbf{e}_{i,j}^\ell=(\alpha_i,\beta_i+\ell(\beta_{i+1}-\beta_i)L^{-1})\}.$$

These completely partition all boundaries of the rectangles.

We are now ready to prove the concentration estimate. Let $G^{\text{low}}_{\lceil nx\rceil,\lceil ny\rceil}$ denote the last passage time in environment determined by c_{low}. Let $\pi_{\max}$ be the maximal path, and let π_k be the segment of the path in the k-th rectangle it visits $n\overline{R}_{i_k,j_k}$.

Now, for each k, π_k will enter and exit $n\overline{R}_{i_k,j_k}$ between two consecutive points of $n\mathscr{D}_{i_k,j_k},n\mathscr{E}_{i_k,j_k}$. We denote by $n\mathbf{z_1}_{i_k,j_k},n\mathbf{z_2}_{i_k,j_k}$ the consecutive points for the entrance and by $n\mathbf{z_1}_{i_{k+1},j_{k+1}},n\mathbf{z_2}_{i_{k+1},j_{k+1}}$ for the exit.

Let $\mathbf{x}$ be a continuous, piecewise linear path from $(0,0)$ to (x,y) so that it crosses through the boundary segments $[\mathbf{z_1}_{i_k,j_k},n\mathbf{z_2}_{i_k,j_k}]$ at some point $\mathbf{x}_k$. Then for L small enough, we have that for some predetermined δ that

$$\Big|\frac{\gamma(\mathbf{z_2}_{i_{k+1},j_{k+2}}-\mathbf{z_1}_{i_k,j_k})}{r_{i_k,j_k}}-\frac{\gamma(\mathbf{x}_{k+1}-\mathbf{x}_k)}{r_{i_k,j_k}}\Big|<\delta.$$

We estimate

$$\begin{aligned}
&\mathbb{P}\{G^{(n)}_{\lceil nx\rceil,\lceil ny\rceil}\ge n\Gamma_c(x,y)+n\theta\}\le\mathbb{P}\{G^{\text{low}}_{\lceil nx\rceil,\lceil ny\rceil}\ge n\Gamma_c(x,y)+n\theta\}\\
&\quad\le\mathbb{P}\Big\{\sum_k G^{\text{low}}_{\pi_k}\ge n\Gamma_{c_{\text{low}}}(x,y)+n(\theta-\varepsilon\gamma(x,y)r_{\min}^{-2})\Big\}\\
&\quad\le\mathbb{P}\Big\{\sum_k G^{\text{low}}_{\lceil n\mathbf{z_1}_{i_k,j_k}\rceil,\lceil n\mathbf{z_2}_{i_{k+1},j_{k+1}}\rceil}\ge n\Gamma_{c_{\text{low}}}(x,y)+n(\theta-\varepsilon\gamma(x,y)r_{\min}^{-2})\Big\}
\end{aligned}$$

$$\leq \mathbb{P}\Big\{ \sum_k G^{\text{low}}_{\lceil n\mathbf{z_1}_{i_k,j_k}\rceil, \lceil n\mathbf{z_2}_{i_{k+1},j_{k+1}}\rceil} \geq n \sum_k \frac{\gamma(\mathbf{x}_{k+1}-\mathbf{x}_k)}{r_{i_k,j_k}} + n(\theta - \varepsilon\gamma(x,y)r_{\min}^{-2}) \Big\}$$

$$\leq \mathbb{P}\Big\{ \sum_k G^{\text{low}}_{\lceil n\mathbf{z_1}_{i_k,j_k}\rceil, \lceil n\mathbf{z_2}_{i_{k+1},j_{k+1}}\rceil} \geq n \sum_k \frac{\gamma(\mathbf{z_2}_{i_{k+1},j_{k+2}} - \mathbf{z_1}_{i_k,j_k})}{r_{i_k,j_k}} + n(\theta - \varepsilon\gamma(x,y)r_{\min}^{-2} - K^2\delta) \Big\}$$

$$\leq \sum_k \mathbb{P}\Big\{ G^{\text{low}}_{\lceil n\mathbf{z_1}_{i_k,j_k}\rceil, \lceil n\mathbf{z_2}_{i_{k+1},j_{k+1}}\rceil} \geq n \frac{\gamma(\mathbf{z_2}_{i_{k+1},j_{k+2}} - \mathbf{z_1}_{i_k,j_k})}{r_{i_k,j_k}} + nK^{-2}(\theta - \varepsilon\gamma(x,y)r_{\min}^{-2} - K^2\delta) \Big\}. \quad (4.16)$$

Because $G^{\text{low}}_{\lceil n\mathbf{z_1}_{i_k,j_k}\rceil, \lceil n\mathbf{z_2}_{i_{k+1},j_{k+1}}\rceil}$ is a last passage time in a homogeneous environment, with limiting shape given by $\frac{\gamma(\mathbf{z_2}_{i_{k+1},j_{k+2}} - \mathbf{z_1}_{i_k,j_k})}{r_{i_k,j_k}}$, the probabilities in the sum above are (upper tail) large deviation probabilities if and only if $\theta - \varepsilon\gamma(x,y)r_{\min}^{-2} - K^2\delta$ can be made positive. This can be achieved when ε is small enough so that $\varepsilon\gamma(x,y)r_{\min}^{-2} < \theta/3$ and then we reduce δ as much as necessary so that $K^2\delta = K^2(\varepsilon)\delta < \theta/3$.

When we guarantee this, we may apply Theorem 4.2 in [34]; this a large deviation principle which gives an exponential concentration inequality for passage times in a homogeneous rate r environment, namely for any $\eta > 0$, we can find a positive $c = c(\eta)$ so that

$$\mathbb{P}\{G_{\lceil nx\rceil,\lceil ny\rceil} > nr^{-1}\gamma(x,y) + n\eta\} \leq e^{-c(\eta)n}.$$

Apply this to each term in the sum (4.16) for $\eta = \theta - \varepsilon\gamma(x,y)r_{\min}^{-2} - K^2\delta$ to finally obtain

$$\mathbb{P}\{G^{(n)}_{\lceil nx\rceil,\lceil ny\rceil} \geq n\Gamma_c(x,y) + n\theta\} \leq Ae^{-\kappa_{\theta,\varepsilon}n}.\square$$

The final approximation before the proof of the main theorem is the limiting time constant in any piecewise constant environment.

Proposition 4.7. *Let $c(s,t)$ be a piecewise constant speed function satisfying assumption 2.2, with a set of discontinuity curves $\{h_i\}_i$ satisfying Assumption 2.1. Let $\mathbf{u} = (x,y) \in \mathbb{R}_2^+$. Then the following law of large numbers holds*

$$\lim_{n\to\infty} \frac{1}{n} G^{(n)}_{\lceil n\mathbf{u}\rceil} = \Gamma_c(\mathbf{u}), \quad \mathbb{P} - a.s. \quad (4.17)$$

Proof of Proposition 4.7. Fix $\mathbf{u} = (x,y) \in \mathbb{R}_+^2$ and consider any admissible path $\mathbf{x} \in \mathcal{H}(x,y)$, viewed as a curve $s \in [0,1] \mapsto \mathbf{x}(s) = (x_1(s), x_2(s))$. Recall the definition of $I(\mathbf{x})$ from (2.3) and remember that $\Gamma = \sup_{\mathbf{x}\in\mathcal{H}(x,y)} I(\mathbf{x})$.

Before proceeding with the technicalities, we highlight the intuition and main approximation idea. The most used technique in literature to prove this kind of limit is to find an upper and lower bound for the microscopic last passage time and then show that they tend to the same macroscopic last passage time in the limit $n \to \infty$. For the lower bound we use the superadditivity property of the microscopic last passage time, and any path acts as a lower bound. For the upper bound we have to construct a particular path which will represent an upper bound for the microscopic last passage time, while approximating the macroscopic limit after scaling its weight by n.

For this, we first partition the rectangle $R_{0,(x,y)} = [0,x] \times [0,y]$ in a very specific way so the following conditions are all satisfied.

1. Isolate the finitely many points of intersection of the discontinuity curves in squares of size δ, where δ will be sufficiently small.
2. Isolate the finitely many points on strictly increasing h_i for which $h_i'(s) = 0$ or $h_i'(s)$ is not defined, in squares of size δ.

Call the collection of these squares by $\mathcal{I}_\delta = \{I_i\}_{1\leq i\leq Q}$. This include points of intersections with the boundary of $R_{0,(x,y)}$. It is fine if these squares overlap, as long as all these problematic points are in their interior.

Away from $\mathcal{I}_\delta$, the discontinuity curves are isolated so that for all curves we can partition each curve h_i finely enough so that for a given tolerance η,

1. Rectangles $R_{h_i(x_j),h_i(x_{j+1})}$ only contain the discontinuity curve h_i. Each rectangle now satisfies Assumption (1) of Lemma 4.3.
2. Assumption (3) in Lemma 4.3 holds for any rectangle $R_{h_i(x_j),h_i(x_{j+1})}$. Assumption (2) of Lemma 4.3 is automatically satisfied away from $\mathcal{I}_\delta$.

3. Since $c(x,y)$ is piecewise constant, any vertical or horizontal discontinuity curves h_i belong to the boundary of the rectangle that agrees with the value of $c(x,y)$ on these curves. Away from $\mathcal{I}_\delta$ these rectangles can be chosen small enough so that $c(x,y)$ is constant on them.
4. If there are discontinuity segments h_i on the north or east boundary of $R_{0,(x,y)}$, we allow degenerate rectangles $R_{h_i(x_j),h_i(x_{j+1})}$ which are linear segments of $\partial R_{0,(x,y)}$.

Call the collection of these rectangles that cover curve h_i by $\mathcal{J}_{h_i,\eta} = \{R_{i,j} = R_{h_i(x_j),h_i(x_{j+1})}\}_j$.

Separating out these rectangles touching or containing problematic points or discontinuity curves, leaves us with with a number D of connected regions D_i, for which $c(x,y)$ restricted on them is constant.

Lower Bound: Any macroscopic path $\mathbf{x}$ can be viewed as the concatenation of a finite number of segments $\mathbf{x}_j$ so that each segment belongs either in a constant rate region, or in one of the rectangles $\mathcal{I}_\delta$ or in one of the rectangles $\cup_i \mathcal{J}_{h_i,\eta}$. Write

$$\mathbf{x}(s) = \sum_{k=1}^{Q} \mathbf{x}(s)\mathbb{1}\{\mathbf{x}(s) \in I_k\} + \sum_{k,\ell} \mathbf{x}(s)\mathbb{1}\{\mathbf{x}(s) \in R_{k,\ell}\} + \sum_{k=1}^{D} \mathbf{x}(s)\mathbb{1}\{\mathbf{x}(s) \in D_k\}.$$

Refine the partition further, so that if $\mathbf{x} : (s_i, s_{i+1}) \to \mathbb{R}^2$ satisfies $\mathbf{x} \subseteq D_k$, then the open rectangle $R_{\mathbf{x}(s_i),\mathbf{x}(s_{i+1})} \subseteq D_k$.

Let $(x_1(s), x_2(s))$ a parametrization of the path $\mathbf{x}$. Partition the interval $[0,1]$ into $\mathcal{P} = \{0 = s_0 < s_1 < s_2 < \ldots < s_K = 1\}$ so that the path segment $\mathbf{x} : [s_i, s_{i+1}] \to \mathbb{R}^2$ belongs to exactly one I_k, $R_{k,\ell}$, or D_k. Note that $I(\mathbf{x}) = \sum_{i=0}^{K-1} \int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\, ds$. The constant $K = K_{\delta,\eta}$ is the total number of different regions the path touches.

We bound each contribution separately:

(1) $\mathbf{x} : (s_i, s_{i+1}) \to \mathbb{R}^2$, $\mathbf{x} \subseteq I_k$. Then, at most,

$$\int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))} < C\delta.$$

Then for all n large enough

$$\Big|\frac{G_{\lceil n\mathbf{x}(s_i)\rceil,\lfloor n\mathbf{x}(s_{i+1})\rfloor}}{n} - \int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\, ds\Big| < C\delta,$$

since also passage times in these rectangles are bounded by $Cn\delta$.

(2) $\mathbf{x} : (s_i, s_{i+1}) \to \mathbb{R}^2$, $\mathbf{x} \subseteq D_k$, where D_k is the homogeneous region of rate r_k. Fix a small $\theta_1 > 0$. Then for all n large enough, by the concentration estimates in [34]

$$\frac{G_{\lceil n\mathbf{x}(s_k)\rceil,\lfloor n\mathbf{x}(s_{k+1})\rfloor}}{n} > \frac{\gamma\big(\mathbf{x}(s_{k+1}) - \mathbf{x}(s_k)\big)}{r_k} - \theta_1 > \int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\, ds - \theta_1.$$

(3) $\mathbf{x} : (s_i, s_{i+1}) \to \mathbb{R}^2$, $\mathbf{x} \subseteq R_{k,\ell}$. Define

$$s_- = \inf\{s \in [s_i, s_{i+1}] : \mathbf{x}(s) - h_k = 0\}, \quad s_+ = \sup\{s \in [s_i, s_{i+1}] : \mathbf{x}(s) - h_k = 0\}.$$

In words, $\mathbf{x}(s_-)$ and $\mathbf{x}(s_+)$ are the points of first and last intersection of $\mathbf{x}$ with h_k in the rectangle $R_{k,\ell}$. Before $\mathbf{x}(s_-)$ and after $\mathbf{x}(s_+)$, $\mathbf{x}$ stays in a constant-rate region, in this rectangle. Between $\mathbf{x}(s_-)$ and $\mathbf{x}(s_+)$, $\mathbf{x}$ touches the discontinuity curve. This rectangle has two constant-rate regions. Denote the smallest one of those by r_{low}.

We bound in the case where the discontinuity curve in the rectangle is increasing. If it is decreasing, $s_- = s_+$, and the argument simplifies since the path $\mathbf{x}$ only intersects the discontinuity at a single point.

Let $G^{(n),\mathcal{N}(\mathbf{x},\varepsilon)}_{\lceil n\mathbf{x}(s)\rceil,\lfloor n\mathbf{x}(t)\rfloor}$ denote the passage time from $\lceil n\mathbf{x}(s)\rceil$ to $\lfloor n\mathbf{x}(t)\rfloor$, subject to the constraint that paths stay in the strip $n\mathcal{N}(\mathbf{x},\varepsilon)$. We assume ε is small enough so that the speed function stays constant on $n\mathcal{N}(\mathbf{x},\varepsilon) \cap R(\lceil n\mathbf{x}(s)\rceil, \lfloor n\mathbf{x}(t)\rfloor)$ except possibly at an $O(\varepsilon)$ region near the beginning and endpoints of the rectangle.

$$\begin{aligned}\frac{G^{(n)}_{\lceil n\mathbf{x}(s_i)\rceil,\lfloor n\mathbf{x}(s_{i+1})\rfloor}}{n} &\geq \frac{G^{(n),\mathcal{N}(\mathbf{x},\varepsilon)}_{\lceil n\mathbf{x}(s_i)\rceil,\lfloor n\mathbf{x}(s_-)\rfloor}}{n} + \frac{G^{(n)}_{\lceil n\mathbf{x}(s_-)\rceil,\lfloor n\mathbf{x}(s_+)\rfloor}}{n} + \frac{G^{(n),\mathcal{N}(\mathbf{x},\varepsilon)}_{\lceil n\mathbf{x}(s_+)\rceil,\lfloor n\mathbf{x}(s_{i+1})\rfloor}}{n}\\ &\geq \int_{s_i}^{s_-} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\, ds - \theta_1 + \frac{\gamma\big(\mathbf{x}(s_-) - \mathbf{x}(s_+)\big)}{r_{\text{low}}} - C_{k,\ell}\,\text{length}(h_k \cap R_{k,\ell})\eta\end{aligned}$$

$$+\int_{s_+}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds - \theta_1 - O(\varepsilon) \tag{4.18}$$

$$\geq \int_{s_i}^{s_-} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds + \int_{s_-}^{s_+} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds + \int_{s_+}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds$$

$$- 2\theta_1 - C_{k,\ell}\text{length}(h_k \cap R_{k,\ell})\eta - O(\varepsilon)$$

$$= \int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds - 2\theta_1 - C_{k,\ell}\text{length}(h_k \cap R_{k,\ell})\eta - O(\varepsilon). \tag{4.19}$$

Line (4.18) follows from Lemma 4.2 for some $\theta_1 > 0$ and n large enough. The line before last follows because either $c(\mathbf{x}(s_k))$ is the largest rate in $R_{i,j}$ or, if it is the smallest of the two, we use Lemma 4.3. The fact that these estimates hold for all large n follows from a Borel-Cantelli argument and the large deviation estimates, as seen in the proof of Lemma 4.1. Constants $C_{k,\ell}$ are the constants given in Lemma 4.3, that show up in bound (4.4). They are all bounded above by some constant $\tilde{C}_\delta$ (which also depends on x, y), since all points where the derivative of increasing h_i is 0 or undefined are isolated in cubes of side δ.

We are now in a position to bound, for all n large enough

$$G^{(n)}_{\lceil nx\rceil,\lceil ny\rceil} \geq \sum_{i=0}^{K_{\delta,\eta}-1} G^{(n)}_{\lceil n\mathbf{x}(s_i)\rceil,\lfloor n\mathbf{x}(s_{i+1})\rfloor}$$

$$\geq n \sum_{i=0}^{K_{\delta,\eta}-1} \int_{s_i}^{s_{i+1}} \frac{\gamma(\mathbf{x}'(s))}{c(\mathbf{x}(s))}\,ds - 3K_{\delta,\eta}n(\theta_1 + O(\varepsilon)) - \tilde{C}\delta n - n\eta\tilde{C}_\delta \sum_{i=1}^{Q} \text{length}(h_i).$$

Divide by n, and take the $\underline{\lim}$ as $n \to \infty$ to obtain

$$\varliminf_{n\to\infty} \frac{G^{(n)}_{\lceil nx\rceil,\lceil ny\rceil}}{n} \geq I(\mathbf{x}) - 3K_{\delta,\eta}(\theta_1 + O(\varepsilon)) - C\delta - C_\delta\eta - O(\varepsilon). \tag{4.20}$$

As the quantifiers go to 0, $K_{\delta,\eta}$ and C_δ blow up, so we first send θ_1 to 0 and $\varepsilon \to 0$. After that send $\eta \to 0$ and finally $\delta \to 0$ to obtain that for an arbitrary $\mathbf{x} \in \mathcal{H}(x, y)$,

$$\varliminf_{n\to\infty} \frac{G^{(n)}_{\lceil nx\rceil,\lceil ny\rceil}}{n} \geq I(\mathbf{x}).$$

A supremum over the class $\mathcal{H}(x, y)$ in the right hand-side of the display above gives

$$\varliminf_{n\to\infty} \frac{G^{(n)}_{\lceil nx\rceil,\lceil ny\rceil}}{n} \geq \Gamma_c(x, y). \tag{4.21}$$

Upper bound: For the upper bound we first partition $[0, x] \times [0, y]$ into rectangles, so that it is a refinement of the partition used for the lower bound: This way conditions (1)-(2) are satisfied and all rectangles in $\cup_i \mathcal{J}_{h_i,\eta}$ and $\mathcal{I}_\delta$ are part of this partition. Outside of the union of $\cup_i \mathcal{J}_{h_i,\eta}$ and $\mathcal{I}_\delta$, only the regions of constant rate remain. Divide each one of the constant region into rectangles, of side no longer than $\delta_1 > 0$ and assume $\delta_1 < \delta$.

Enumerate the rectangles in the two-dimensional partition by $Q_{i,j} = [(x_i, x_{i+1}] \times (y_j, y_{j+1}]$ and their total number is $N_{\eta,\delta,\delta_1} < \infty$. Just a small point of caution: In principle the boundaries of $Q_{i,j}$ can contain vertical or horizontal discontinuity lines of $c(x, y)$ and those should be included on the rectangle that agrees with value of c on their boundaries. It is without loss of generality, and to simplify the already heavy notation that we assume here that potential discontinuity curves are on the north or east boundary of $Q_{i,j}$.

Now, for any $n \in \mathbb{N}$ define the environment according to $c(x, y)$ and consider the maximizing path $(0, 0)$ to $(\lceil nx\rceil, \lceil ny\rceil)$ which we denote by $\pi^{\max}_{0,(\lceil nx\rceil,\lceil ny\rceil)}$. The path can be written as a finite concatenation of sub-paths

$$\pi^{\max}_{0,(\lceil nx\rceil,\lceil ny\rceil)} = \sum_{(x_i,y_j)} \pi_{\lceil nQ_{i,j}\rceil}$$

where $\pi_{\lceil nQ_{i,j}\rceil}$ is the segment of the path in the rectangle $(\lceil nx_i\rceil, \lceil nx_{i+1}\rceil] \times (\lceil ny_j\rceil, \lceil ny_{j+1}\rceil]$. Some of these segments will be empty.

We partition the sides of each rectangle $Q_{i,j}$ further: Fix a $\delta_2 > 0$ and define partitions

$$\mathscr{P}_{e_1,(i,j)} = \{\mathbf{h}_k^{(i,j)} = (x_i, y_j) + k\delta_2 e_1\}_{0 \le k \le \frac{x_{i+1}-x_i}{\delta_2}}, \quad \mathscr{P}_{e_2,(i,j)} = \{\mathbf{v}_k^{(i,j)} = (x_i, y_j) + k\delta_2 e_2\}_{0 \le k \le \frac{y_{i+1}-y_i}{\delta_2}}.$$

These completely define a partition of the boundaries $Q_{i,j}$. Now, the entry point of $\pi_{\lceil nQ_{i,j}\rceil}$ into $nQ_{i,j}$ will be between two consecutive partition points, say $\mathbf{a}_k^{(i,j)} \le \mathbf{a}_{k+1}^{(i,j)}$ and its exit point will be between $\mathbf{b}_\ell^{(i,j)} \le \mathbf{b}_{\ell+1}^{(i,j)}$. Note that exit point of one rectangle will be the entry point in an adjacent one, and all these points belong to some partition $\mathscr{P}_{e_k,(i,j)}$. If it so happens and the path enters (or exits) from one of the macroscopic partition points, we set $\mathbf{a}_k^{(i,j)} = \mathbf{a}_{k+1}^{(i,j)}$ (equiv. $\mathbf{b}_\ell^{(i,j)} = \mathbf{b}_{\ell+1}^{(i,j)}$).

When the environment in $Q_{i,j}$ is constant $r_{i,j}$, we have the bound

$$\begin{aligned} G^{(n)}_{\lceil nQ_{i,j}\rceil}(\pi) &= \sum_{v \in \pi_{\lceil nQ_{i,j}\rceil}} \tau_v^{(n)} \le G^{(n)}_{n\mathbf{a}_k^{(i,j)}, n\mathbf{b}_{\ell+1}^{(i,j)}} \le n\Big(\frac{\gamma(\mathbf{b}_{\ell+1}^{(i,j)} - \mathbf{a}_k^{(i,j)})}{r_{i,j}} + \theta_1\Big) \\ &\le n\Big(\frac{\gamma(\mathbf{b}_{\ell}^{(i,j)} - \mathbf{a}_{k+1}^{(i,j)})}{r_{i,j}} + C_{i,j}\omega_\gamma(\delta_2) + \theta_1\Big). \end{aligned} \tag{4.22}$$

The second-to-last inequality follows by Theorem 4.2 in [34], for n large enough.

When $c(s,t)$ on $Q_{i,j}$ takes two values, r_1, r_2 separated by a curve h, we bound as follows. First fix a tolerance ε and find $\delta_3 > 0$ so that we may define a continuous speed function $c_{\delta_3,h}(s,t)$ as in Lemma 3.3, with the property $c_{\delta_3,h}(s,t) \le c(s,t)$ and

$$\sup_{\mathbf{a}_k, \mathbf{b}_\ell} (\Gamma_{c_{\delta_3,h}}(\mathbf{a}_k, \mathbf{b}_\ell) - \Gamma_c(\mathbf{a}_k, \mathbf{b}_\ell)) < \varepsilon. \tag{4.23}$$

Then,

$$\begin{aligned} G^{(n)}_{\lceil nQ_{i,j}\rceil}(\pi) &= \sum_{v \in \pi_{\lfloor nQ_{i,j}\rfloor}} \tau_v^{(n)} \le G^{(c_{\delta_3,h})}_{n\mathbf{a}_k^{(i,j)}, n\mathbf{b}_{\ell+1}^{(i,j)}} \\ &\le n\big(\Gamma_{c_{\delta_3,h}}(\mathbf{a}_k^{(i,j)}, \mathbf{b}_{\ell+1}^{(i,j)}) + \theta_1\big) \text{ by a Borel-Cantelli argument and Lemma 4.6}, \\ &\le n\big(\Gamma_{c_{\delta_3,h}}(\mathbf{a}_{k+1}^{(i,j)}, \mathbf{b}_{\ell}^{(i,j)}) + \omega_{\Gamma_c}(2\delta_2) + \theta_1\big) \text{ by Theorem 2.4}, \qquad (4.24) \\ &\le n\big(\Gamma_{c}(\mathbf{a}_{k+1}^{(i,j)}, \mathbf{b}_{\ell}^{(i,j)}) + \varepsilon + \omega_{\Gamma_c}(2\delta_2) + \theta_1\big) \text{ by equation 4.23}. \qquad (4.25) \end{aligned}$$

Using the estimates (4.22) and (4.25), we have total upper bound for the passage time

$$\begin{aligned} G^{(n)}_{\lceil nx\rceil,\lceil ny\rceil} &= \sum_{(i,j)} G^{(n)}_{\lceil nQ_{i,j}\rceil}(\pi) \\ &\le n\sum_{(i,j)} \Gamma_c(\mathbf{a}_{k+1}^{(i,j)}, \mathbf{b}_\ell^{(i,j)}) + nN_{\eta,\delta,\delta_1}(\max_{(i,j)} C_{i,j}\omega_\gamma(\delta_2) + \theta_1 + \varepsilon + \omega_{\Gamma_c}(2\delta_2)) + nC|\mathcal{I}_\delta|\delta \\ &\le n(\Gamma_c(x,y) + N_{\eta,\delta,\delta_1}(\max_{(i,j)} C_{i,j}\omega_\gamma(\delta_2) + \theta_1 + \varepsilon + \omega_{\Gamma_c}(2\delta_2)) + C|\mathcal{I}_\delta|\delta) \end{aligned}$$

The last line follows from superadditivity of Γ. To finish the bound, divide by n and take the $\overline{\lim}$ as $n \to \infty$. Then, let $\delta_2 \to 0$. This will result to finer $\mathscr{P}_{e_k,(i,j)}$ partitions, but by modulating δ_3 we can still keep estimate (4.23) with the same ε. Then let θ_1 and ε tend to 0. These are independent of the other quantifiers η, δ_1 and δ. Finally send $\delta \to 0$. □

Proof of Theorem 2.6. Fix (x,y) and fix an $\epsilon > 0$. It is always possible to find piecewise strictly positive constant functions c_1 and c_2 such that $||c_1 - c_2||_\infty \le \epsilon$ that definitely have the same discontinuity curves as the function c (but perhaps more). On $[0,x] \times [0,y]$ we can further impose $c_1(x,y) \le c(x,y) \le c_2(x,y)$, by defining each c_i on smaller rectangles.

When the weights in (1.4) are defined via the speed function c_i we write G^i for last passage time and Γ_{c_i} for their limits. A coupling using common exponential variables $\{\tau_{i,j}\}$ gives

$$G^{1,(n)}_{\lceil nx\rceil,\lceil ny\rceil} \ge G^{(n)}_{\lceil nx\rceil,\lceil ny\rceil} \ge G^{2,(n)}_{\lceil nx\rceil,\lceil ny\rceil}.$$

Letting $r_{\min} > 0$ denote a lower bound for $c(x, y)$ in the rectangle $[0, x] \times [0, y]$. Then we bound for any $\mathbf{x} \in \mathcal{H}$:

$$
\begin{aligned}
0 &\leq \int_0^1 \left\{ \frac{\gamma(\mathbf{x}'(s))}{c_1(x_1(s), x_2(s))} - \frac{\gamma(\mathbf{x}'(s))}{c_2(x_1(s), x_2(s))} \right\} ds \\
&= \int_0^1 \frac{\gamma(\mathbf{x}'(s))(c_2(x_1(s), x_2(s)) - c_1(x_1(s), x_2(s)))}{c_1(x_1(s), x_2(s))c_2(x_1(s), x_2(s))} ds \leq \epsilon \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c_1^2(x_1(s), x_2(s))} ds \\
&\leq \epsilon r_{\min}^{-2} \gamma(x, y).
\end{aligned}
$$

As the inequality is uniform across $\mathbf{x}$, the bound extends to the suprema

$$0 \leq \Gamma_{c_1}(x, y) - \Gamma_{c_2}(x, y) \leq C(x, y)\epsilon.$$

From Proposition 4.7 we know that the Γ_{c_i} are the limits for $G^{i,(n)}$. To obtain Theorem 2.6, let $\varepsilon \to 0$. □

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